Exercises for Nuclear Astrophysics I
WS 2011/12 - Prof. Shawn Bishop
Sheet 4

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Problems will be discussed in exercises on 11/17/2011 at 13:00.

6 Mean Molecular Weights

This problem serves to familiarize yourself with the important concept of mean molecular weights.

(1) Calculate and interpret the values of μ under the following circumstances:

- (a) All hydrogen ($X = 1$, $Y = 0$), fully ionized
- (b) All helium ($X = 0$, $Y = 1$), partial ionization (average charge state: He^{1+})
- (c) All helium ($X = 0$, $Y = 1$), fully ionized
- (d) All metals ($X = 0$, $Y = 0$), fully ionized

(2) Calculate the mean molecular weight per electron μ_e for completely ionized conditions of

- (a) all hydrogen
- (b) all helium, also find cases where electron degenerate helium is important in stars
- (c) $X = Z = 0.5$

(3) Assuming conditions under which the assumption

$$\mu \approx \frac{2}{1 + 3X + 0.5Y} \tag{1}$$

is valid, calculate the rate of change of the mean molecular weight

- (a) with respect to the metal content Z, holding the hydrogen fraction constant $\left(\frac{\partial\mu}{\partial Z}\right)_X$
- (b) with respect to the helium content Y, holding the metal fraction Z constant $\left(\frac{\partial\mu}{\partial Y}\right)_Z$
- (c) Use the results to interpret the change of μ and the pressure P during hydrogen and helium burning.

7 Main sequence stars and polytropes

Assume a star with the properties $M = 30\ M_\odot$, $R = 6.6\ R_\odot$, $X = 0.7$ and $Y = 0.3$ has been observed.

(a) Calculate the central pressure P_c.

(b) Calculate the central temperature T_c.*

Hint: You need to use the parameterized plot of β from lecture 3 - slide 22.

8 Stellar energetics

In this exercise, we want to derive a stability criterium for stars.

(a) Show that the gravitational energy $\Omega = -G \int \frac{M_r}{r} dM_r$ can also be written as $\Omega = -3 \int P dV$.
Hint: Use the hydrostatic equilibrium equation.

(b) We now introduce the mass-averaged temperature $< T >:= \frac{1}{M} \int T dM_r$. Show that for an ideal gas:

$$M < T >= -\frac{1}{3}\Omega \frac{\mu}{N_A k} \tag{2}$$

(c) Now use the mass-specific heat capacity $c_v = \frac{\partial}{\partial M_r}\left(\frac{\partial U}{\partial T}\right)_v$ to show that for a thin, spherical shell of the star, where you can assume $c_v = \text{const.}$:

$$U = c_v \int T dM_r \tag{3}$$

(d) Use the results from (b) and (c) to show that the total energy of the star $W = \Omega + U$ can be written as:

$$W = -(3\gamma - 4)U \tag{4}$$

where $\gamma = c_p/c_v$.

(e) What would a value of $\gamma < \frac{4}{3}$ mean for the stability of the star? Do you know stars/conditions where this can happen? Hint: Think about very massive stars.

9 Stellar Mass Squared

Use the hydrostatic equilibrium equation

$$\frac{dP}{dr} = -G\frac{M_r}{r^2}\rho(r) \tag{5}$$

and the definition of the infinitesimal mass of a spherical shell

$$dM_r = 4\pi r^2 \rho(r) dr \tag{6}$$

to show, that the mass-squared of a spherical fraction of the star enclosed within the radius r_i can be written as

$$M^2(r_i) = \frac{8\pi}{G}\left[4\int_0^{r_i} P(r)r^3 dr - P(r_i)r_i^4\right] \tag{7}$$

Also give the relation for the total mass-squared of the star $M^2(R)$.