Exercises for Nuclear Astrophysics I
WS 2011/12 - Prof. Shawn Bishop
Sheet 3

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Problems will be discussed in exercises on 11/10/2011 at 13:00.

5 Lane-Emden equation and stellar modelling

5.1 Polytrope with $n = 0$

Consider a stellar model with finite radius R and constant density profile $\rho(r) = \rho_c$ (see excercise sheet 1). This corresponds to a polytrope of index $n = 0$.

(a) **Ritter's First Integral**

Show, that in this case, the solution of the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\phi_n}{d\xi}) = -\phi_n^n \tag{1}$$

with the boundary conditions $\phi_n(0) = 1$ and $\phi_n'(0) = 0$ at the center, and $\phi_n(\xi_1) = 0$ at the surface is parabolic:

$$\phi_0(\xi) = 1 - \frac{\xi^2}{6} \tag{2}$$

(b) **Another lower bound on the central pressure**

In Problem 3, a lower bound on the central pressure was derived. A stroger lower limit can be obtained using the constant density model. Consider the Lagrangian form of the hydrostatic equilibrium equation

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4} \tag{3}$$

derived in Problem 2.3. Eliminate the variable r and integrate the equation to obtain the pressure

$$P = P_c\left[1 - \left(\frac{M_r}{M}\right)^{2/3}\right] \tag{4}$$

Use this to give a lower limit on the central pressure P_c. Note that although this is a stronger lower limit than that of Problem 3, we did not actually prove that this is really a lower limit here.

5.2 Polytrope with $n = 1$: Ritter's Second Integral

Consider a polytrope with index $n = 1$. Show, that in this case, the Lane-Emden Equation takes the form of the sperical Bessel differential equation

$$r^2\frac{d^2R(r)}{dr^2} + 2r\frac{R(r)}{r} + (k^2r^2 - m(m+1))R(r) = 0 \tag{5}$$

with the general solution

$$R(r) = A \cdot j_n(kr) + B \cdot n_m(kr) \tag{6}$$

where $j_n(kr)$ and $n_m(kr)$ are the spherical Bessel functions of first and second order respectively. Then, derive the solution $\phi_1(\xi)$ by applying the boundary condition $\phi_1(0) = 1$ to the general solution (6)

Hint: Find the coefficients k and m first, by comparing equations (1) and (5).

5.3 General calculations for Polytropes

(a) **Density ratio**

Show that the ratio of mean to central density for any polytropic index n is given by

$$\frac{\overline{\rho}}{\rho_c} = -3\left(\frac{\phi'}{\xi}\right)_{\xi_*} \tag{7}$$

where $\phi' := \frac{d\phi}{d\xi}$ and $\xi_* = \xi_1$ is the first zero of ϕ and thus describes the surface of the star.

(b) **Central Pressure**

Show, that the central pressure can be written as

$$P_c = \frac{1}{4\pi(n+1)(\phi'_n)^2_{\xi_*}} \frac{GM^2}{R^4} \tag{8}$$