Exercises for Nuclear Astrophysics I - WS 2010/11
Sheet 2

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Problems with increased difficulty are marked with a (*)

3 A lower bound for the central pressure

The goal of this exercise is to derive a (relatively weak) limit on the central pressure of stars. Consider the function

$$F(r) = P(r) + \frac{GM_r^2}{8\pi r^4} \tag{1}$$

where M_r is the mass enclosed in a sphere of radius r.

1. Show that $F(r)$ decreases outward with increasing r by showing $\frac{dF}{dr} < 0$.
2. Show that $\lim_{r\to 0} F(r) = P_c$, where P_c is the central pressure. Hint: You will need to use l'Hospital's rule twice.
3. Calculate the resulting lower limit on P_c in solar units.

4 The Pressure Integral

In the lecture, the Pressure Integral was derived in the form

$$P = \frac{1}{3}\int_0^\infty p v_p n(p) dp \tag{2}$$

where $n(p)$ is the normalized distribution of the magnitude of momentum and v_p is the velocity corresponding to momentum p.

1. **Ideal, non-degenerate, non-relativistic gas ***

   Show that for an ideal gas with the Maxwell-Boltzmann distribution

   $$n(p) = \frac{4\pi p^2 N}{(2\pi mkT)^{3/2}} \exp\left(-\frac{p^2}{2mkT}\right) \tag{3}$$

   the pressure integral yields the familiar result $P = NkT$, where N is the number of particles per unit volume and $v_p = \frac{p}{m}$. Hint: Consider the function $I(a) = \int_0^\infty \exp(-ap^2)dp = \frac{1}{2}\sqrt{\frac{\pi}{a}}$.

2. **Completely degenerate, non-relativistic gas**

   It is shown in the lecture, that the pressure P_e of a completely electron-degenerate, non-relativistic gas is given by

   $$P_e = \frac{h^2}{20m}\left(\frac{3}{\pi}\right)^{2/3} n_e^{5/3} \tag{4}$$

   (a) Verify this result by evaluating the pressure integral for the corresponding momentum distribution. Hint: The momentum distribution is very simple, assume every electron occupies a phase space volume of h^3.

   (b) Find a relation between the electron density n_e, the density ρ and the mean molecular weight per electron μ_e. Set equal the pressures of non-degenerate and degenerate ideal gas and show, that an estimate for the set-in of electron degeneracy can be given by

   $$\frac{\rho}{\mu_e} \approx 2.4 \cdot 10^{-8} \cdot T^{3/2} \frac{g}{cm^3} \tag{5}$$

   (c) Use the equation obtained in (b) to confirm the non-degeneracy of the center of our sun using $\rho = 150\ \frac{g}{cm^3}$, $T = 15.7 \cdot 10^6\ K$, $X = 75\%$ and $Y = 24\%$.

(d) Use the equation obtained in (b) to confirm the degeneracy of a white dwarf (You will learn more about these stars in the lecture) assuming a temperature of $T \approx 10^6\ K$, $\rho \approx 10^6\ \frac{g}{cm^3}$ and a composition of 50% carbon and 50% oxygen.

3. **Completely degenerate, relativistic gas**

(a) Starting from the maximum momentum of the degenerate gas $p_m = \left(\frac{3h^3}{8\pi} n_e\right)^{1/3}$, calculate $\frac{\rho}{\mu_e}$ where $p_m c = m_0 c^2$ (This is just an arbitrary condition for the set-in of relativistic conditions, you could also use e.g. $p_m c = 2 m_0 c^2$.)

(b) Derive a relation for $v_p(p)$ from elementary relativistics and use it to show, that the pressure integral yields

$$P_{rel,deg} = \frac{8\pi m^4 c^5}{3h^3} \left(\frac{\sinh^3 \theta_m \cosh \theta_m}{4} - \frac{3 \sinh(2\theta_m)}{16} + \frac{3\theta_m}{8} \right) \tag{6}$$

with the substitution $\sinh \theta = \frac{p}{mc}$.

(c) Examine the non-relativistic ($\frac{p}{mc} \to 0$) and the completely relativistic ($\frac{p}{mc} \to \infty$) limit of (6) by applying suitable Taylor expansions.

5 The Partition Function

Consider the partition function:

$$Z(\tau) = \sum_s \exp(-\frac{\epsilon_s}{\tau}) \tag{7}$$

where $\tau = kT$ and ϵ_s is the energy of a state s. Show that the average energy U of a system can be written as:

$$U = \tau^2 \frac{\partial \log Z}{\partial \tau} \tag{8}$$