Nuclear Astrophysics

Lecture 1

Tues. April 24, 2012

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Resonant Reaction Rate:

$$r_{12} = \left(\frac{8}{\pi\mu}\right)^{1/2} \frac{N_1 N_2}{1+\delta_{12}} \tau^{-3/2} \int_0^\infty E\sigma(v) \exp\left(-\frac{E}{\tau}\right) dE$$

$$\sigma = \frac{\pi}{k^2} \frac{\Gamma_a \Gamma_b}{(E-E_r)^2 + \Gamma^2/4} = \frac{\pi\hbar^2}{2\mu E} \frac{\Gamma_a \Gamma_b}{(E-E_r)^2 + \Gamma^2/4}$$

Assume that both widths are small (total width is narrow) and they are constant (very weak energy dependence). Sub the above cross section into the integral, treating the Γ terms as constants:

$$r_{12} = \left(\frac{8}{\pi\mu}\right)^{1/2} \frac{\pi\hbar^2}{2\mu} \frac{N_1 N_2}{1+\delta_{12}} \tau^{-3/2} e^{-E_r/\tau} \frac{\Gamma_a \Gamma_b}{\Gamma} 2 \underbrace{\int_0^\infty \frac{\Gamma/2}{(E-E_r)^2 + \Gamma^2/4} dE}_{=\pi}$$

Resonant Reaction Rate (Single Resonance):

$$r_{12} = \left(\frac{2\pi}{\mu\tau}\right)^{3/2} \hbar^2 \frac{N_1 N_2}{1+\delta_{12}} \frac{2J_r+1}{(2J_1+1)(2J_2+1)} \frac{\Gamma_a \Gamma_b}{\Gamma} e^{-E_r/\tau}$$

We have added the spin-statistical factors (now) to account for the fact that our theory up to this point has only considered spin zero particles (in entrance channel and exit channel) in the reaction.

More generally, for particles with spin, we have to multiply the cross section by the spin-statistical factor to account for the different permutations of spin alignments that are allowed.

J_r is the spin (intrinsic) of the resonance.
J_1 " " beam particle.
J_2 " " target particle.

We define $\omega\gamma \equiv \frac{2J_r+1}{(2J_1+1)(2J_2+1)} \frac{\Gamma_a \Gamma_b}{\Gamma}$ as the *resonance strength.*

In the event that several resonances can contribute to the rate, then we have:

$$r_{12} = \left(\frac{2\pi}{\mu\tau}\right)^{3/2} \hbar^2 \frac{N_1 N_2}{1+\delta_{12}} \sum_i (\omega\gamma)_i e^{-E_{r_i}/\tau}$$

Where the sum extends over all resonances that can contribute to the rate, and where each resonance has its own resonance energy and resonance strength.

Inverse Reaction Rate

Let's start with Fermi's Golden Rule: $T_{a \to b} = \frac{2\pi}{\hbar} |H_{ab}|^2 \rho(E_b)$

Where ρ_b is the density of *final* states and: $|H_{ab}| \equiv \langle b | H_p | a \rangle$

Here, H_p is the "perturbing" component of the Hamiltonian that generates transitions.

Let's consider the ratio of the forward and inverse rates: $\frac{T_{a \to b}}{T_{b \to a}} = \left| \frac{H_{ab}}{H_{ba}} \right|^2 \frac{\rho(E_b)}{\rho(E_a)}$

Now, if $H_p = H_p^*$ then: $|H_{ab}|^2 = |H_{ba}|^2$ (exercise for you)

Our transition rate ratio is now: $\frac{T_{a \to b}}{T_{b \to a}} = \frac{\rho(E_b)}{\rho(E_a)}$

From the definition of cross section, the transition rate of particles scattered into any solid angle element is just: "Flux times cross section"

$$\frac{T_{a\to b}}{T_{b\to a}} = \frac{v_a\, d\sigma_{a\to b}}{v_b\, d\sigma_{b\to a}} = \frac{\rho(E_b)}{\rho(E_a)}$$

Recall from Lecture 2, of last semester that the density of states for a free particle in final state a is given by:

$$\rho(p_a) \equiv \frac{1}{V}\frac{dN_a}{dp_a} = g_a \frac{p_a^2}{\pi^2\hbar^3}$$

And from Chain Rule: $$\frac{dN}{dp} = \frac{dN}{dE}\frac{dE}{dp}$$

$$\Rightarrow \rho(p) \equiv \frac{1}{V}\frac{dN}{dp} = \frac{1}{V}\frac{dN}{dE}\frac{dE}{dp} = \rho(E)\frac{dE}{dp}$$

Collecting everything, we now have:

$$g_a\, v_a\, p_a^2\, \frac{dp_a}{dE_a}\, d\sigma_{a\to b} = g_b\, v_b\, p_b^2\, \frac{dp_b}{dE_b}\, d\sigma_{b\to a}$$

Almost there: $$g_a\, v_a\, p_a^2\, \frac{dp_a}{dE_a}\, d\sigma_{a\rightarrow b} = g_b\, v_b\, p_b^2\, \frac{dp_b}{dE_b}\, d\sigma_{b\rightarrow a}$$

If particles are massive in either entrance or exit channels, "a" and "b", then:

$$p^2 = 2\mu E \;\Rightarrow\; \frac{dp}{dE} = \frac{\mu}{p}$$

$$\boxed{g_a\, \mu_a E_a\, d\sigma_{a\rightarrow b} = g_b\, \mu_b E_b\, d\sigma_{b\rightarrow a}}$$

All particles massive in entrance & exit

On the other hand, if one of the particles in exit channel (say, particle "4") is a gamma, then a bit more work.

For a massive particle-gamma combination, the center of mass lies at the massive particle. (Why?) This means, then, that the relative momentum of this system is simply the photon momentum (Why?).

From relativity: $$E_\gamma^2 = p_\gamma^2/c^2 \;\Rightarrow\; \frac{dp}{dE} = c^2 E_\gamma/p_\gamma$$

And so, for the reaction case: $1 + 2 \rightleftharpoons 3 + \gamma$

We have the relation:

$$g_a \, 2\mu_a E_a \, d\sigma_{a\rightarrow b} = g_b \, c^2 E_\gamma^2 \, d\sigma_{b\rightarrow a}$$

We finally have a relation between the photodisintegration cross section and the charged particle cross section

$$\sigma_{3\gamma}(E_\gamma) = c^2 \frac{(2J_1+1)(2J_2+1)}{2(2J_3+1)} 2\mu_{12} \frac{E_{12}}{E_\gamma^2} \, \sigma(E_{12})$$

From Lecture 6, the photodisintegration reaction rate was found to be:

$$r_{3\gamma} = \frac{8\pi N_3}{h^3 c^2} \int_Q^\infty \sigma_{3\gamma}(E_\gamma) \frac{E_\gamma^2}{\exp(E_\gamma/\tau) - 1} dE_\gamma$$

Recall that $E_\gamma = E_{12} + Q_{12\rightarrow 3\gamma}$.

Making the substitution for photodisintegration cross section and changing variable, we have now:

$$r_{3\gamma} = \frac{(2J_1+1)(2J_2+1)}{(2J_3+1)} \frac{8\pi N_3}{h^3} \int_0^\infty \mu_{12} E_{12} \frac{\sigma(E_{12})}{\exp([E_{12}+Q]/\tau)-1} dE_{12}$$

Now, $\tau = kT = 8.67 \times 10^{-11} T \text{ MeV/K}$. At explosive temperatures between 10^6 to 10^9 Kelvin, this is a small number. And the argument of the exponential is, at minimum, just the Q-value. Q-values are of the order of a few MeV. So, the exponential term is large – very large – compared to unity. We therefore can write:

$$r_{3\gamma} = \frac{(2J_1+1)(2J_2+1)}{(2J_3+1)} \frac{8\pi \mu_{12} N_3}{h^3} \exp(-Q/\tau) \int_0^\infty E_{12}\sigma(E_{12}) \exp(-E_{12}/\tau) dE_{12}$$

From Lecture 6, the forward reaction rate is given by:

$$r_{12} = \left(\frac{8}{\pi \mu_{12}}\right)^{1/2} N_1 N_2 \tau^{-3/2} \int_0^\infty E_{12}\sigma(E_{12}) \exp\left(-E_{12}/\tau\right) dE_{12}$$

Ratio of inverse photodisintegration rate to the forward charged particle rate is:

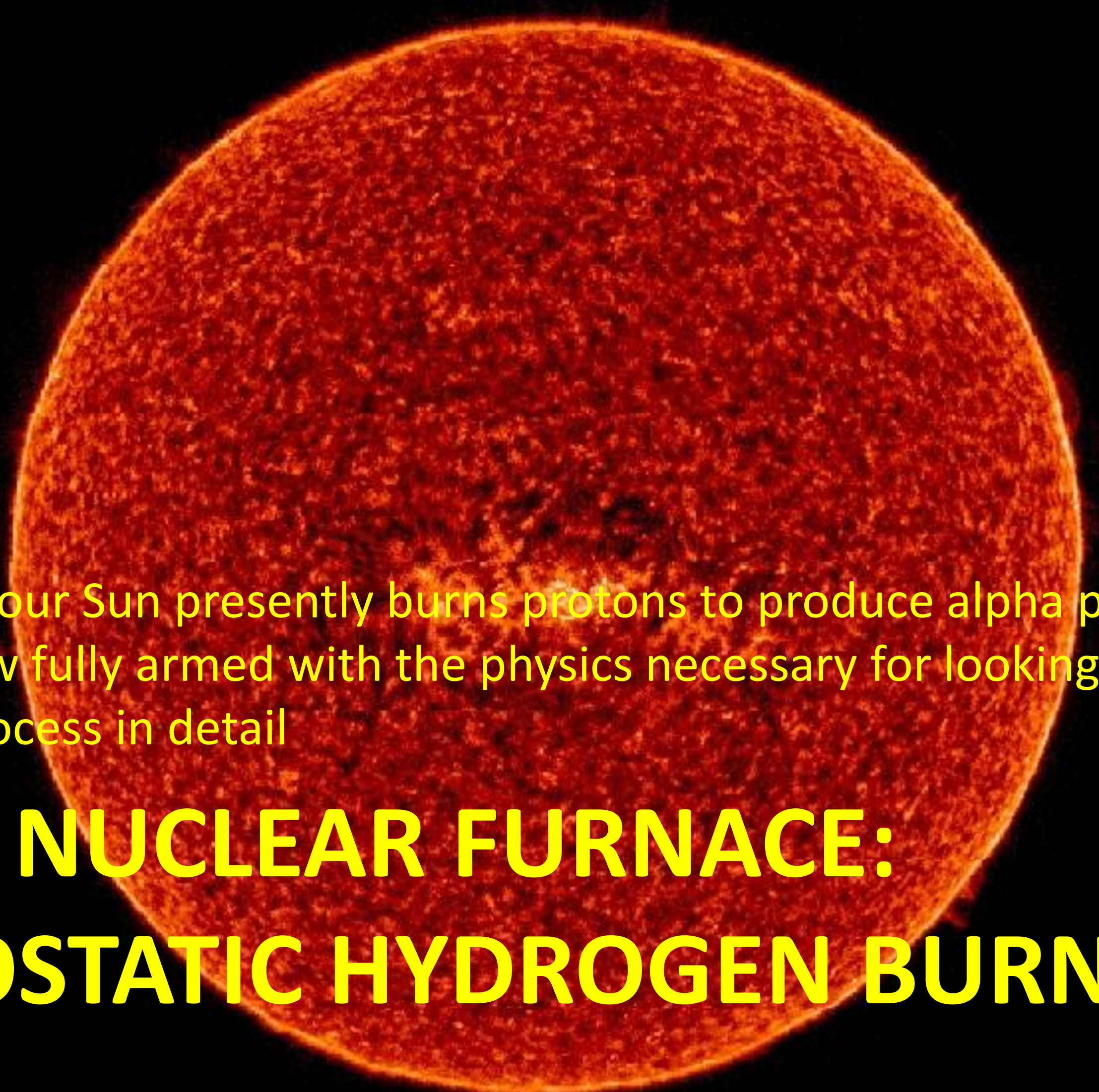
$$\frac{r_{3\gamma}}{r_{12}} = \frac{g_1 g_2}{g_3} \frac{N_3}{N_1 N_2} \left(\frac{2\pi}{h^2} \right)^{3/2} (\mu_{12}\tau)^{3/2} \exp(-Q_{12\rightarrow 3\gamma}/\tau)$$

Note: In equilibrium, with the forward and inverse rates equal to each other, the above equation is exactly the Saha Equation.

Exercise for you. Show that the ratio of rates when all particles are massive is:

$$\frac{r_{34}}{r_{12}} = \frac{g_1 g_2}{g_3 g_4} \frac{N_3 N_4}{N_1 N_2} \left(\frac{\mu_{34}}{\mu_{12}} \right)^{1/2} \exp(-Q_{12\rightarrow 34}/\tau)$$

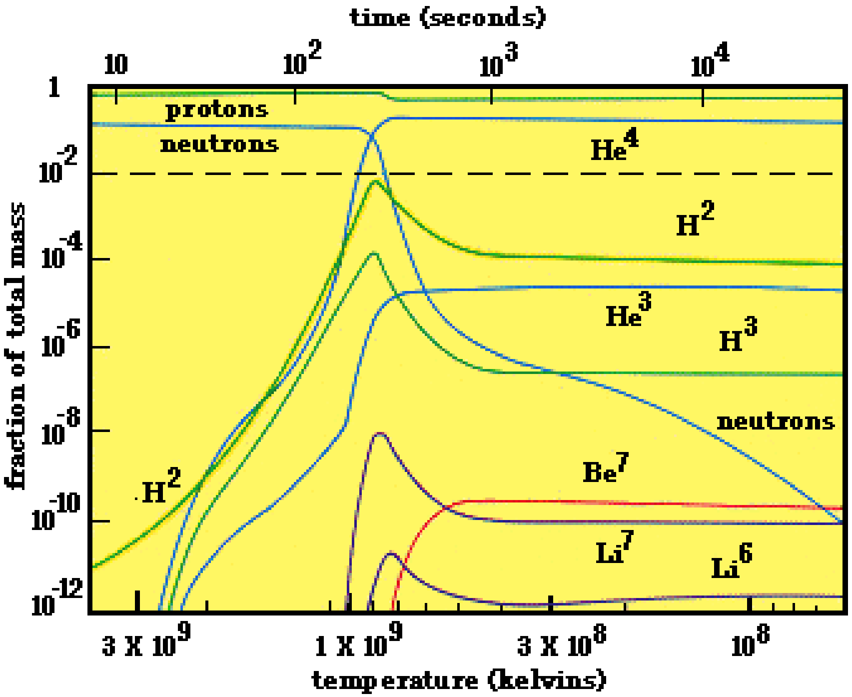
These two results will be required for future lectures on the nuclear processes occurring within extreme phenomena such as Supernova and X-Ray bursts. **Know** them well!



In its core, our Sun presently burns protons to produce alpha particles. We are now fully armed with the physics necessary for looking at this burning process in detail

SOLAR NUCLEAR FURNACE: HYDROSTATIC HYDROGEN BURNING

After primordial BBNS, the dominant isotopic constituents are protons and alpha particles. The first generation of stars had to condense from gas consisting of these isotopes (and a tiny fraction of others).



$$p + p \to {}^2\text{He} \to p + p$$

$$p + {}^4\text{He} \to {}^5\text{Li} \to p + {}^4\text{He}$$

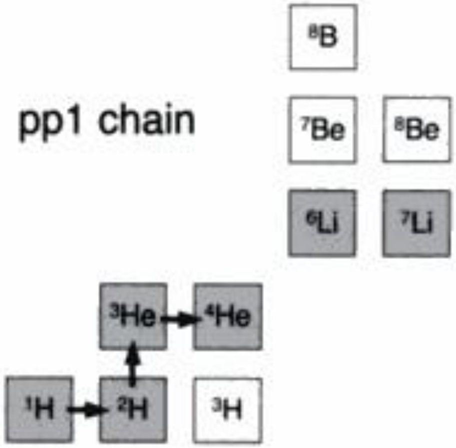
$${}^4\text{He} + {}^4\text{He} \to {}^8\text{Be} \to {}^4\text{He} + {}^4\text{He}$$

All unstable

Must start with something else. P + D will not work: there is not sufficient abundance of D to sustain the burning times of a star. What else?

$$p + p \to D + \beta^+ + \nu$$

This is the next most sensible reaction to consider. It's a weak interaction, so it will be very slow compared to the strong interaction reactions above. But we know there is plenty of hydrogen in the star to make it feasible.

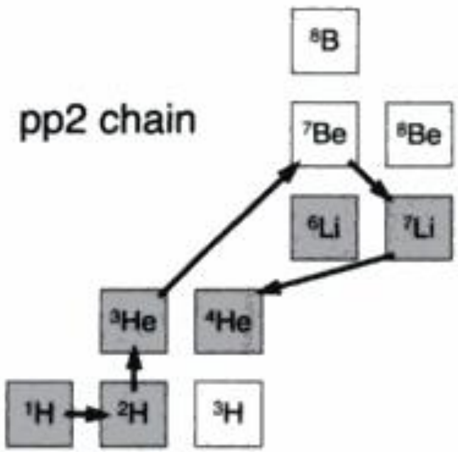


$$p + p \rightarrow D + \beta^+ + \nu$$

$$p + D \rightarrow {}^3\mathrm{He} + \gamma$$

Life on Earth possible

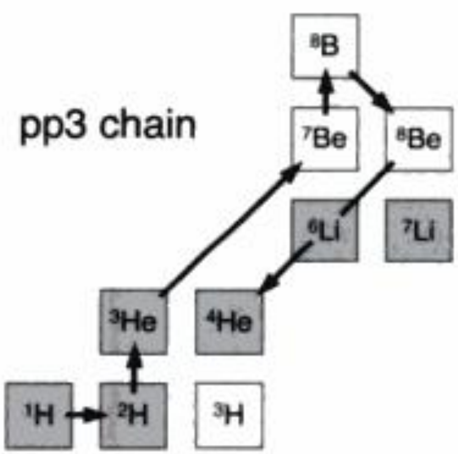
$$^3\mathrm{He} + {}^3\mathrm{He} \rightarrow \alpha + 2p$$



$$^3\mathrm{He} + {}^4\mathrm{He} \rightarrow {}^7\mathrm{Be} + \gamma$$

$$^7\mathrm{Be} + \beta^- \rightarrow {}^7\mathrm{Li} + \nu$$

$$^7\mathrm{Li} + p \rightarrow \alpha + \alpha$$



$$^7\mathrm{Be} + p \rightarrow {}^8\mathrm{B} + \gamma$$

$$^8\mathrm{B} \rightarrow \beta^+ + \nu + {}^8\mathrm{Be}$$

$$^8\mathrm{Be} \rightarrow \alpha + \alpha$$

Proton-Proton-Chain

$p + p \rightarrow d + e^{+} + \nu$

$p + d \rightarrow {}^{3}He + \gamma$

86% | 14%

86%:

${}^{3}He + {}^{3}He \rightarrow {}^{4}He + 2p$

PP-I

$Q_{eff} = 26.20$ MeV

14%:

${}^{3}He + {}^{4}He \rightarrow {}^{7}Be + \gamma$

99.7% | 0.3%

99.7%:

${}^{7}Be + e^{-} \rightarrow {}^{7}Li + \nu$

${}^{7}Li + p \rightarrow 2\ {}^{4}He$

PP-II

$Q_{eff} = 25.66$ MeV

0.3%:

${}^{7}Be + p \rightarrow {}^{8}B + \gamma$

${}^{8}B \rightarrow {}^{8}Be + e^{+} + \nu$

$\downarrow$

$2\ {}^{4}He$

PP-III

$Q_{eff} = 19.17$ MeV

Netto: $4p \rightarrow {}^{4}He + 2e^{+} + 2\nu + Q_{eff}$

Some reminders and definitions before we start:

$$\langle \sigma v \rangle \equiv 4\pi \left(\frac{\mu}{2\pi\tau} \right)^{3/2} \int_0^\infty v^3 \sigma(v) \exp\left(-\frac{\mu v^2}{2\tau} \right) dv$$

$$= \left(\frac{8}{\pi\mu} \right)^{1/2} \tau^{-3/2} \int_0^\infty E\sigma(E) \exp\left(-\frac{E}{\tau} \right) dE$$

Reaction rate formula, between particles 1 and 2 is then:

$$r_{12} = \frac{N_1 N_2}{1 + \delta_{12}} \langle \sigma v \rangle_{12}$$

Lifetime of particle 1 to destruction by particle 2:

$$\tau_2(1) \equiv \frac{1}{N_2 \langle \sigma v \rangle_{12}} = \left(\rho \frac{X_2}{M_2} N_A \langle \sigma v \rangle_{12} \right)^{-1}$$

PPI Chain Abundance Equations:

$$p + p \rightarrow D + \beta^+ + \nu$$

$$p + D \rightarrow {}^3\mathrm{He} + \gamma$$

$${}^3\mathrm{He} + {}^3\mathrm{He} \rightarrow \alpha + 2p$$

$$\frac{dD}{dt} = r_{pp} - r_{Dp}$$

$$= \frac{H^2}{2}\langle\sigma v\rangle_{pp} - HD\langle\sigma v\rangle_{pD}$$

$$\frac{d({}^3\mathrm{He})}{dt} = DH\langle\sigma v\rangle_{pD} - 2\frac{({}^3He)^2}{2}\langle\sigma v\rangle_{33}$$

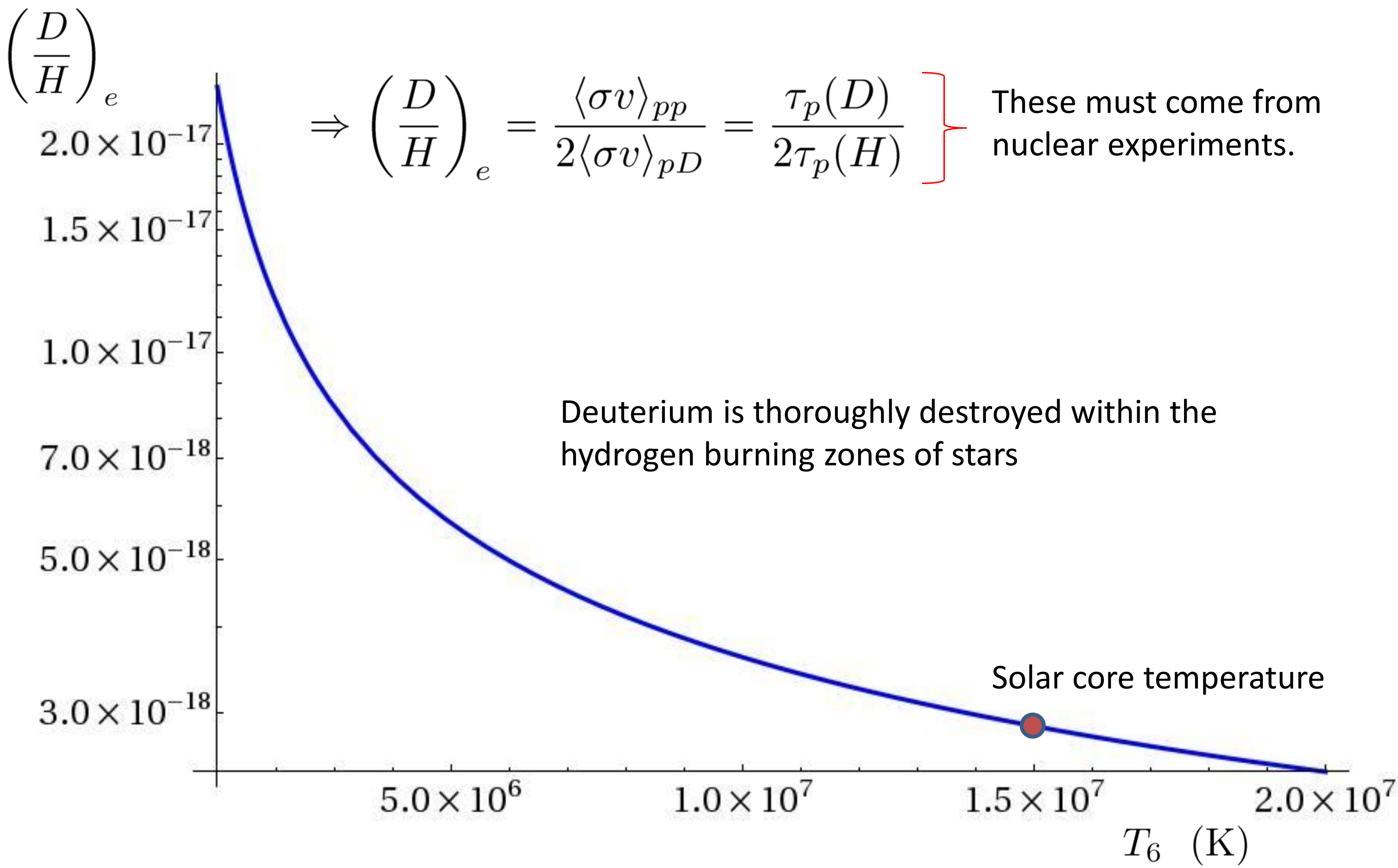
Two ^{3}He destroyed per reaction (left and right sides of equation are coupled)

Now, compared to the p + p rate (weak interaction, so very slow), the p + D rate will be very fast (EM Interaction). Therefore, D should reach an equilibrium abundance.

With D in equilibrium, this means $\frac{dD}{dt} = 0$

$$\Rightarrow \left(\frac{D}{H}\right)_e = \frac{\langle \sigma v \rangle_{pp}}{2\langle \sigma v \rangle_{pD}} = \frac{\tau_p(D)}{2\tau_p(H)}$$

These must come from nuclear experiments.



The lifetime (destruction rate) of deuterium is so short (fast) compared to its production through the p + p reaction rate, that we can safely assume that the hydrogen abundance in the star changes *negligibly* during the time in which deuterium reaches its equilibrium.

We can then solve the deuterium abundance as a function of time:

$$\frac{d(D/H)}{dt} = \frac{H}{2}\langle \sigma v \rangle_{pp} - H\left(\frac{D}{H}\right)\langle \sigma v \rangle_{pD}$$

H is now constant!

Transform with: $x = D/H$ $\quad a = \frac{H}{2}\langle \sigma v \rangle_{pp}$ $\quad b = H\langle \sigma v \rangle_{pD}$

Then we have: $\frac{dx}{dt} + bx - a = 0$

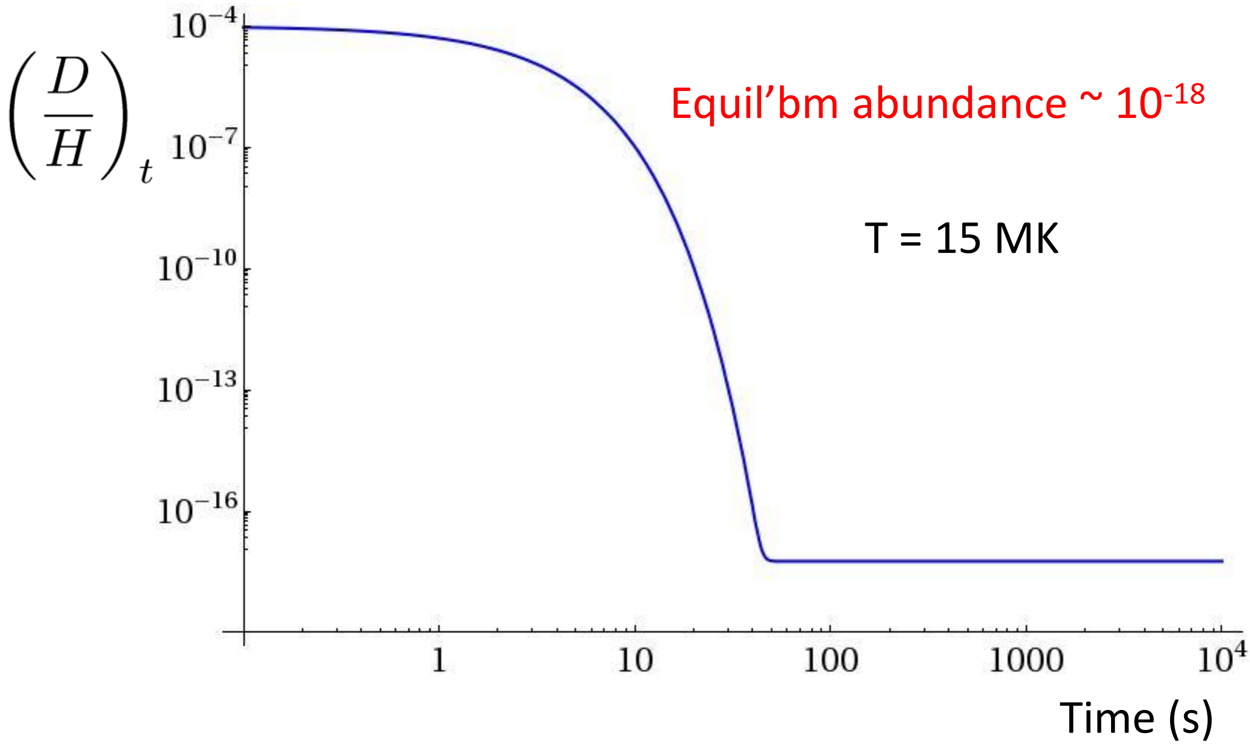
Solve this by multiplying both sides with integrating factor e^{bt}

$$\Rightarrow \frac{d}{dt}\left\{x\, e^{bt}\right\} = -a\, e^{bt}$$

Exercise for you: Take the last equation and work the solution through to the end, under the initial condition that at t = 0 the deuterium fraction is: $(D/H)_{t=0}$.

$$\left(\frac{D}{H}\right)_t = \left(\frac{D}{H}\right)_e - \left[\left(\frac{D}{H}\right)_e - \left(\frac{D}{H}\right)_{t=0}\right] e^{-t/\tau_p(D)}$$

(Recall: $H\langle\sigma v\rangle_{pD} \equiv \dfrac{1}{\tau_p(D)}$)



Plot shows something important: Deuterium achieves equilibrium abundance within less than a few *minutes*! Main sequence stars burn hydrogen for billions of years.

Assumption that deuterium is in equilibrium and rapidly destroyed is justified.

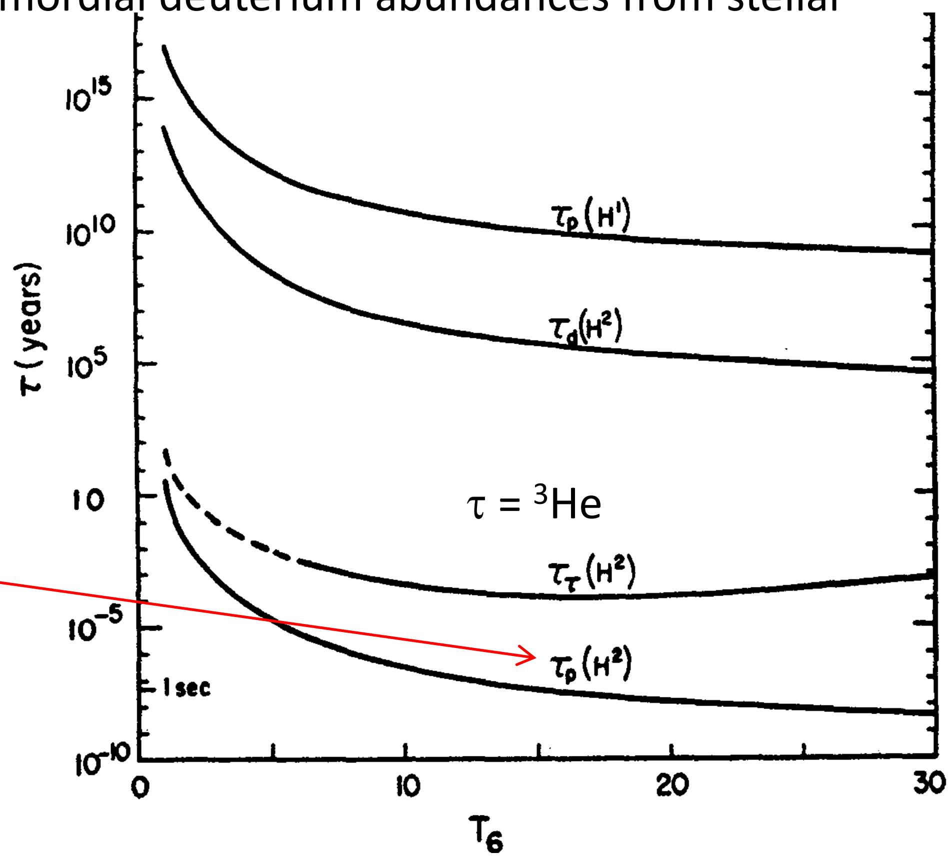
What does this result tell us about stars in the early Universe?

- Primordial deuterium would have been incorporated in the first generations of stars
- We have seen that it burns with hydrogen at a prodigious rate compared to the p + p reaction rate
- Early generation stars that condensed out of the first gas clouds most likely started their fusion burning with the p + D reaction ***not*** the p + p reaction
- ***Dangerous*** to try extracting primordial deuterium abundances from stellar atmosphere Fraunhofer lines!

Lifetimes of species in the PP1 chain. Again: the lifetime of deuterium against destruction by protons is, by far, the smallest

$$X_H = X_{He} = 0.50$$

$$\rho = 100 \text{ g} \cdot \text{cm}^{-3}$$



Moving on now to the ^{3}He abundance: $\frac{d(^3\mathrm{He})}{dt} = DH\langle\sigma v\rangle_{pD} - 2\frac{(^3He)^2}{2}\langle\sigma v\rangle_{33}$

The deuterium is in equilibrium. We can therefore use: $\left(\frac{D}{H}\right)_e = \frac{\langle\sigma v\rangle_{pp}}{2\langle\sigma v\rangle_{pD}} = \frac{\tau_p(D)}{2\tau_p(H)}$

Exercise: Use the equilibrium expression for D/H to write:

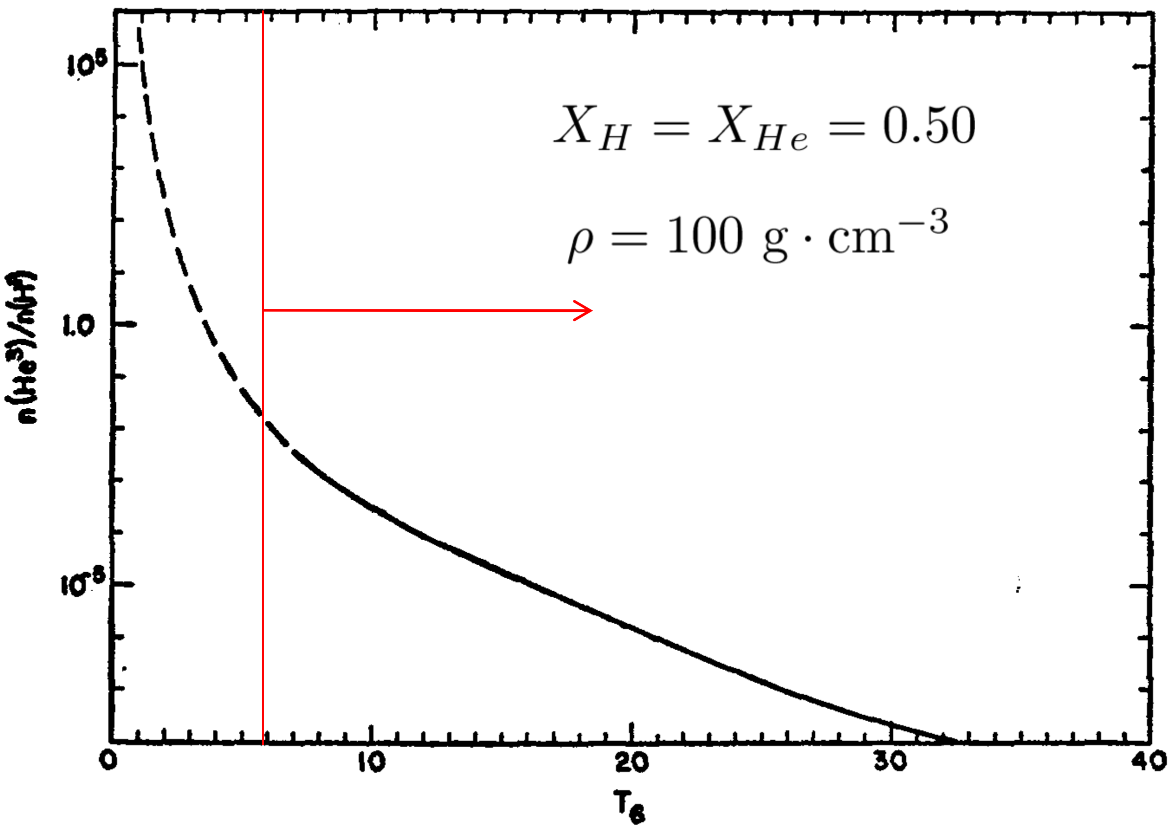
$$\frac{d(^3He/H)}{dt} = \frac{H}{2}\langle\sigma v\rangle_{pp} - H\left(\frac{^3He}{H}\right)^2\langle\sigma v\rangle_{33}$$

Transform this using: $x = (^3He/H)$ $\quad a = (H/2)\langle\sigma v\rangle_{pp}$ $\quad b = H\langle\sigma v\rangle_{33}$

$$\Rightarrow \frac{dx}{dt} = a - bx^2$$

Looks like something involving a trigonometric integral.

^{3}He *equilibrium* abundance ratio to H, as a function of temperature. For temperatures above ~ 6 MK, the ratio is tiny; meaning the H number density is overwhelmingly larger than that of ^{3}He → solve the previous equation treating H as a constant.

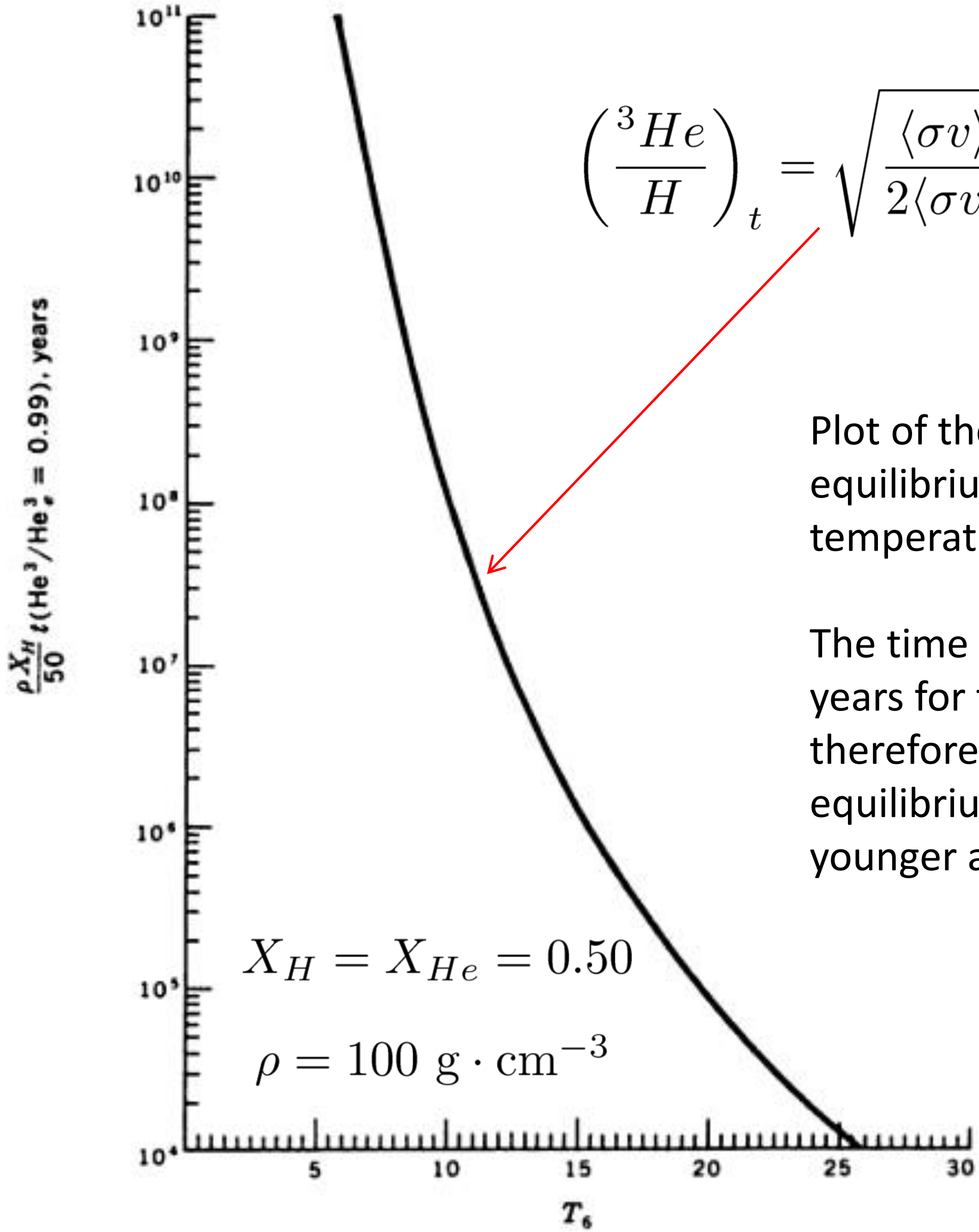


Exercise for you: Solve the previous equation using trigonometric substitution, or a table of integrals, to determine $^3He/H$ as a function of time, with the initial condition that at t = 0, $^3He/H = 0$.

$$\left(\frac{^3He}{H}\right)_t = \sqrt{\frac{\langle\sigma v\rangle_{pp}}{2\langle\sigma v\rangle_{33}}}\tanh\left(t\sqrt{\frac{H}{2}\langle\sigma v\rangle_{pp} H\langle\sigma v\rangle_{33}}\right)$$

$$= \left(\frac{^3He}{H}\right)_e \tanh\left(\frac{t}{[\tau_3(3)]_e}\right)$$

$$\left(\frac{{}^3He}{H}\right)_t = \sqrt{\frac{\langle\sigma v\rangle_{pp}}{2\langle\sigma v\rangle_{33}}}\tanh\left(t\sqrt{\frac{H}{2}\langle\sigma v\rangle_{pp}H\langle\sigma v\rangle_{33}}\right)$$



Plot of the time for ^{3}He/H to reach 99% of its equilibrium abundance as a function of temperature.

The time for ^{3}He to reach equil'm is billions of years for temperatures less than ~ 8 MK. It is therefore dangerous to assume ^{3}He is in equilibrium for standard main sequence stars younger and less massive than our Sun.

Energy generation rate of the PPI Chain:

$$\epsilon_{ppI} = \frac{1}{\rho}(Q_{pp}\, r_{pp} + Q_{pD}\, r_{pD} + Q_{33}\, r_{33})$$

$$= \frac{1}{\rho}\left(Q_{pp}\,\frac{H^2}{2}\langle\sigma v\rangle_{pp} + Q_{pD} HD\langle\sigma v\rangle_{pD} + Q_{33}\frac{^3He}{2}\langle\sigma v\rangle_{33}\right)$$

Recall the results about deuterium (D) abundance changes: D achieves equilibrium within ***minutes***. From page 9, with dD/dt = 0:

(**) $$HD\langle\sigma v\rangle_{pD} = \frac{H^2}{2}\langle\sigma v\rangle_{pp}$$ Substitute into above eqn

When ^{3}He is in equilibrium, d(^{3}He)/dt = 0

Exercise for you: Use the differential equation for , d(^{3}He)/dt = 0 on page 9. Use it to express the $\langle\sigma v\rangle_{33}$ factor in terms of a factor involving $\langle\sigma v\rangle_{pD}$. Then use the equation (**) to express the $\langle\sigma v\rangle_{pD}$ factor in terms of a $\langle\sigma v\rangle_{pD}$ term.

This will let us write the energy generation rate in terms of the p + p reaction rate only.

After you have done the previous exercise, you should find:

$$\epsilon_{ppI} = \frac{H^2}{2\rho}\left(Q_{pp} + Q_{pD} + \frac{Q_{33}}{2}\right)\langle\sigma v\rangle_{pp}$$

$$H = \rho N_A \frac{X_H}{A_H}$$

$$= \frac{N_A^2}{2}\frac{X_H^2}{A_H^2}\left(Q_{pp} + Q_{pD} + \frac{Q_{33}}{2}\right)\langle\sigma v\rangle_{pp}$$

From lecture 11, we know the $\langle\sigma v\rangle_{pp}$ factor can be written in a power-law form with exponent

$$n = (-2 + 3\frac{E_{\rm eff}}{\tau})/3$$

Lec. 11

$$3\frac{E_{\rm eff}}{\tau} = 42.487\left(\frac{Z_1^2 Z_2^2 \mu}{T_6}\right)^{1/3}$$

For p + p, we have for T_6 = 15, n = 3.9

$$\epsilon_{ppI} = \epsilon_{ppI}(T_0)\left(\frac{T}{T_0}\right)^{3.9}$$

$$\frac{dH}{dt} = 2\frac{He^2}{2}\langle\sigma v\rangle_{^3He^3He} - 2\frac{H^2}{2}\langle\sigma v\rangle_{pp} - HD\langle\sigma v\rangle_{pD} - H(^7\mathrm{Be})\langle\sigma v\rangle_{p^7Be} - H(^7\mathrm{Li})\langle\sigma v\rangle_{p^7Li}$$ 1

$$\frac{dD}{dt} = \frac{H^2}{2}\langle\sigma v\rangle_{pp} - HD\langle\sigma v\rangle_{pD}$$ 2

$$\frac{d(^3\mathrm{He})}{dt} = DH\langle\sigma v\rangle_{pD} - 2\frac{(^3He)^2}{2}\langle\sigma v\rangle_{^3He^3He} - (^3He)(^4He)\langle\sigma v\rangle_{\alpha^3He}$$ 3

$$\frac{d(^4\mathrm{He})}{dt} = \frac{(^3He)^2}{2}\langle\sigma v\rangle_{^3He^3He} + 2H(^7\mathrm{Be})\langle\sigma v\rangle_{p^7Be} + 2H(^7\mathrm{Li})\langle\sigma v\rangle_{p^7Li} - (^3He)(^4He)\langle\sigma v\rangle_{\alpha^3He}$$ 4

$$\frac{d(^7\mathrm{Be})}{dt} = (^3He)(^4He)\langle\sigma v\rangle_{\alpha^3He} - H(^7\mathrm{Be})\langle\sigma v\rangle_{p^7Be} - (^7\mathrm{Be})\lambda_{e^7Be}$$ 5

$$\frac{d(^7\mathrm{Li})}{dt} = (^7\mathrm{Be})\lambda_{e^7Be} - H(^7\mathrm{Li})\langle\sigma v\rangle_{p^7Li}$$ 6

Exercise for you:

Take the system of equations from previous page and:

1. Assume that ^{7}Be, ^{7}Li have lifetimes that are short compared to star → both are in equilibrium, so that

$$\frac{d(^7\mathrm{Li} +^7 \mathrm{Be})}{dt} = 0$$

2. Assume ^{3}He has also come to equilibrium in Equation 3:

$$\frac{d(^3He)}{dt} = 0$$

3. Use also Deuterium equilibrium in Eq. 2

4. Use these simplifications to reduce the previous system of equations down to just two differential equations involving the rate of change of hydrogen and alphas.

5. Also find an expression for the **equilibrium abundance** of ^{3}He from the simplification in step 2 and using Eq. 3. Show that your equation makes sense in the limits that H→0 and ^{4}He → 0.

PP Chain Abundance Evolution (No ^{3}He Equilibrium Simplification)