Nuclear Astrophysics

Lecture 9

Thurs. Dec. 22, 2011

Prof. Shawn Bishop, Office 2013,

Ex. 12437

shawn.bishop@ph.tum.de

The 4 Equations of Stellar Structure

$$(A) \quad \frac{dP}{dr} = -G\frac{M(r)}{r^2}\rho(r)$$

$$(B) \quad \frac{dM(r)}{dr} = 4\pi r^2 \rho$$

$$(C) \quad \frac{dL}{dr} = 4\pi\epsilon r^2 \rho$$

$\epsilon =$ Energy generation rate per unit mass of material

$$(D) \quad L(r) = -4\pi r^2 \frac{c}{\rho\bar{\kappa}}\frac{dP_\gamma}{dr}$$

$\bar{\kappa} =$ average opacity coefficient in the material

Return to the Standard (Stellar) Model

The stellar gas is a mixture of photons and Ideal particles. Thus,

$$P_g = n\tau = \frac{N_A \rho}{\mu} kT \qquad P_\gamma = \frac{\pi^2}{45\hbar^3 c^3}\tau^4 = \frac{a}{3}T^4$$

Total Pressure: $P_{tot} = P_g + P_\gamma$

$$a = \frac{\pi^2 k^4}{15\hbar^3 c^3}$$

In thermodynamic equilibrium, these two gases have the same temperature. And let

$P_g = \beta P_{tot}$ and $P_\gamma = (1-\beta)P_{tot}$

Then, we have: $$T = \left(\frac{3}{a}\frac{1-\beta}{\beta}\frac{N_A k \rho}{\mu}\right)^{1/3}$$

$$\Rightarrow P_{tot} = \left(\frac{3}{a}\frac{1-\beta}{\beta^4}\right)^{1/3}\left(\frac{kN_A}{\mu}\right)^{4/3}\rho^{4/3}$$

Polytrope of type 3

Polytrope Solutions

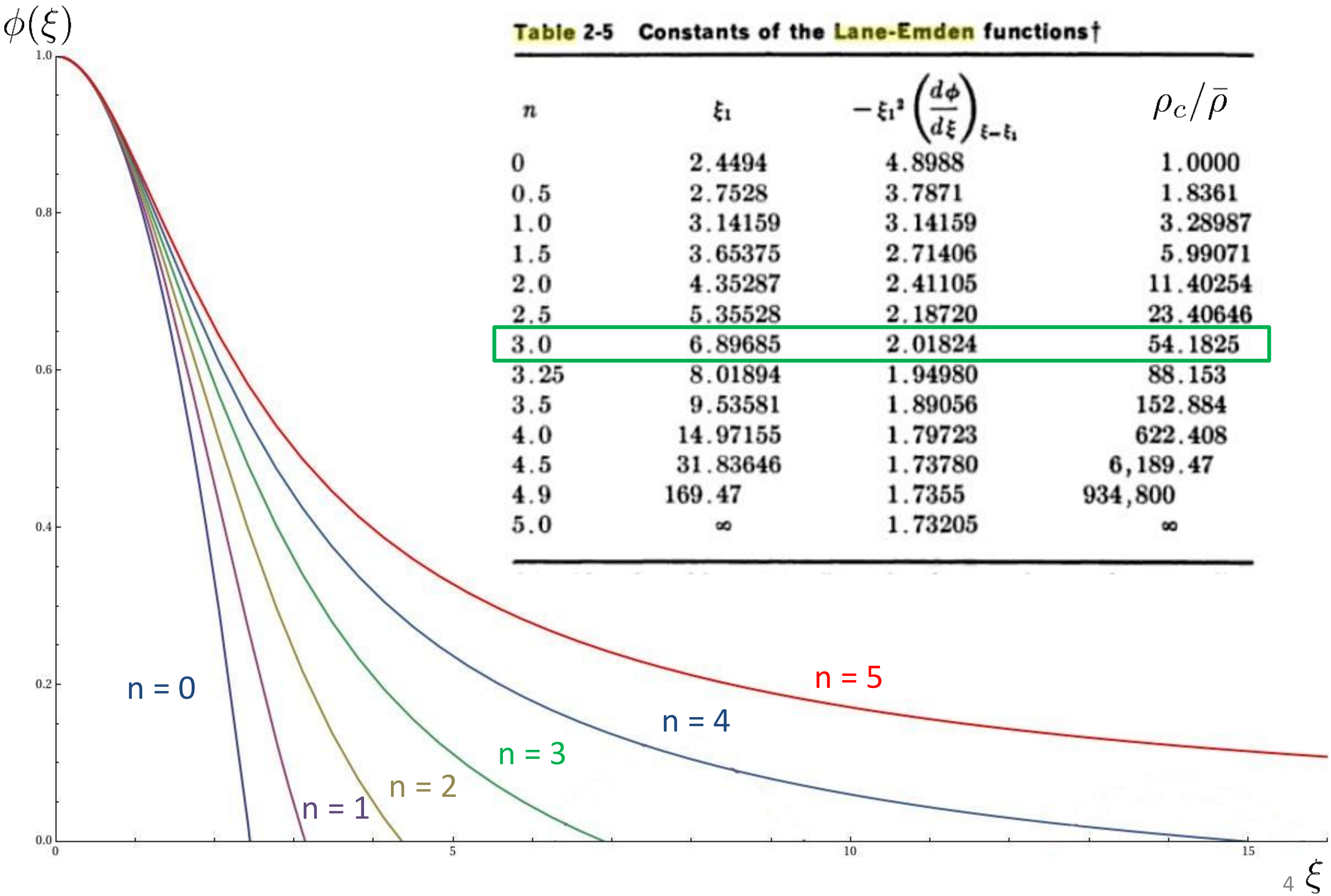


Table 2-5 Constants of the Lane-Emden functions†

n	ξ_1	$-\xi_1^2\left(\frac{d\phi}{d\xi}\right)_{\xi=\xi_1}$	$\rho_c/\bar{\rho}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6,189.47
4.9	169.47	1.7355	934,800
5.0	∞	1.73205	∞

Mass Luminosity Relation

Take the following equations from the 4 Structure Equations:

$$L(r) = -4\pi r^2 \frac{c}{\rho\bar{\kappa}} \frac{dP_\gamma}{dr}$$

$$\frac{dP}{dr} = -G\frac{M(r)}{r^2}\rho(r)$$

And use our friend: $P_\gamma = (1-\beta)P$

$$\Rightarrow \frac{dP_\gamma}{dr} = -G\frac{M(r)}{r^2}(1-\beta)\rho(r)$$

Sub this into first equation above

$$\Rightarrow L(r) = \frac{4\pi c G M_r}{\kappa}(1-\beta)$$

Small for all but the most massive of stars.

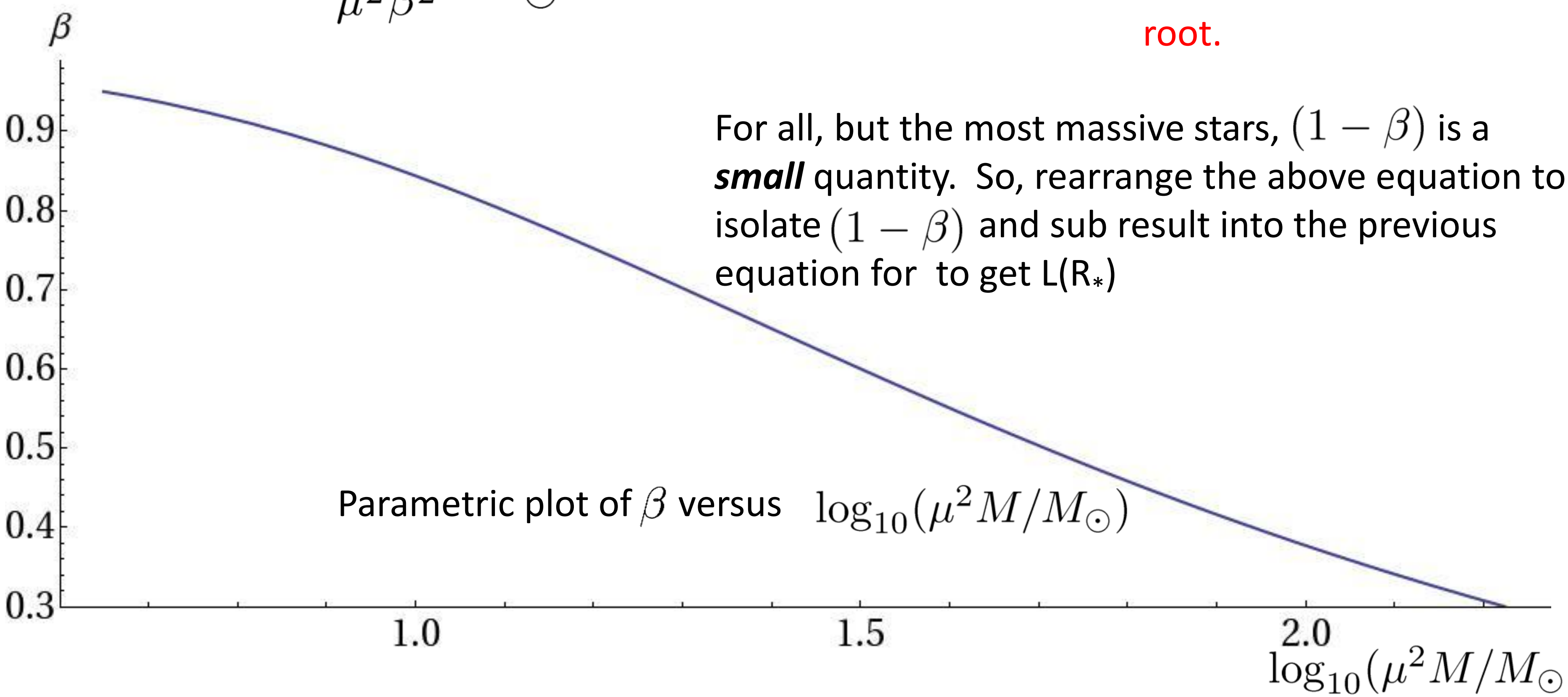
Eddington's Quartic Equation

In 3rd Lecture, pages 13 & 20, it was shown that the stellar mass can be written as,

$$M_* = -\frac{4\pi}{(\pi G)^{3/2}}\left(\frac{3}{a}\right)^{1/2}\frac{\sqrt{1-\beta}}{\beta^2}\left(\frac{kN_A}{\mu}\right)^2\left(\xi^2\frac{d\phi}{d\xi}\right)_{\xi_*}$$

$$= 18.0\frac{\sqrt{1-\beta}}{\mu^2\beta^2}M_\odot$$

Product of the root of ϕ and its 1st derivate at the root.



For all, but the most massive stars, $(1-\beta)$ is a ***small*** quantity. So, rearrange the above equation to isolate $(1-\beta)$ and sub result into the previous equation for to get $L(R_*)$

Parametric plot of β versus $\log_{10}(\mu^2 M/M_\odot)$

After the algebra (you should check, to make sure I'm right ☺), we arrive at the **Mass Luminosity Relation** for Main Sequence stars!

$$L_* = L(R_*) = \frac{\pi^2 caG^4}{12\kappa}\beta^4\left(\frac{\mu}{kN_A}\right)^4\left(\xi^2\frac{d\phi}{d\xi}\right)^{-2}_{\xi_*}M_*^3$$

Numerically: $$L_* = 1.35\times 10^{35}\frac{(\mu\beta)^4\left(\frac{M_*}{M_\odot}\right)^3}{\kappa}\ \mathrm{erg/s}$$

We now have our first theoretical prediction of the relationship between two observable properties of stars.

The Luminosity of Main Sequence Stars (H-burning, hydrostatic, up to ~15 solar masses) should be proportional to the 3rd power of the stellar mass. (first order result, we can do better).

To do better, we have to deal with that annoying opacity, κ .

Total Opacity of Solar Composition Material

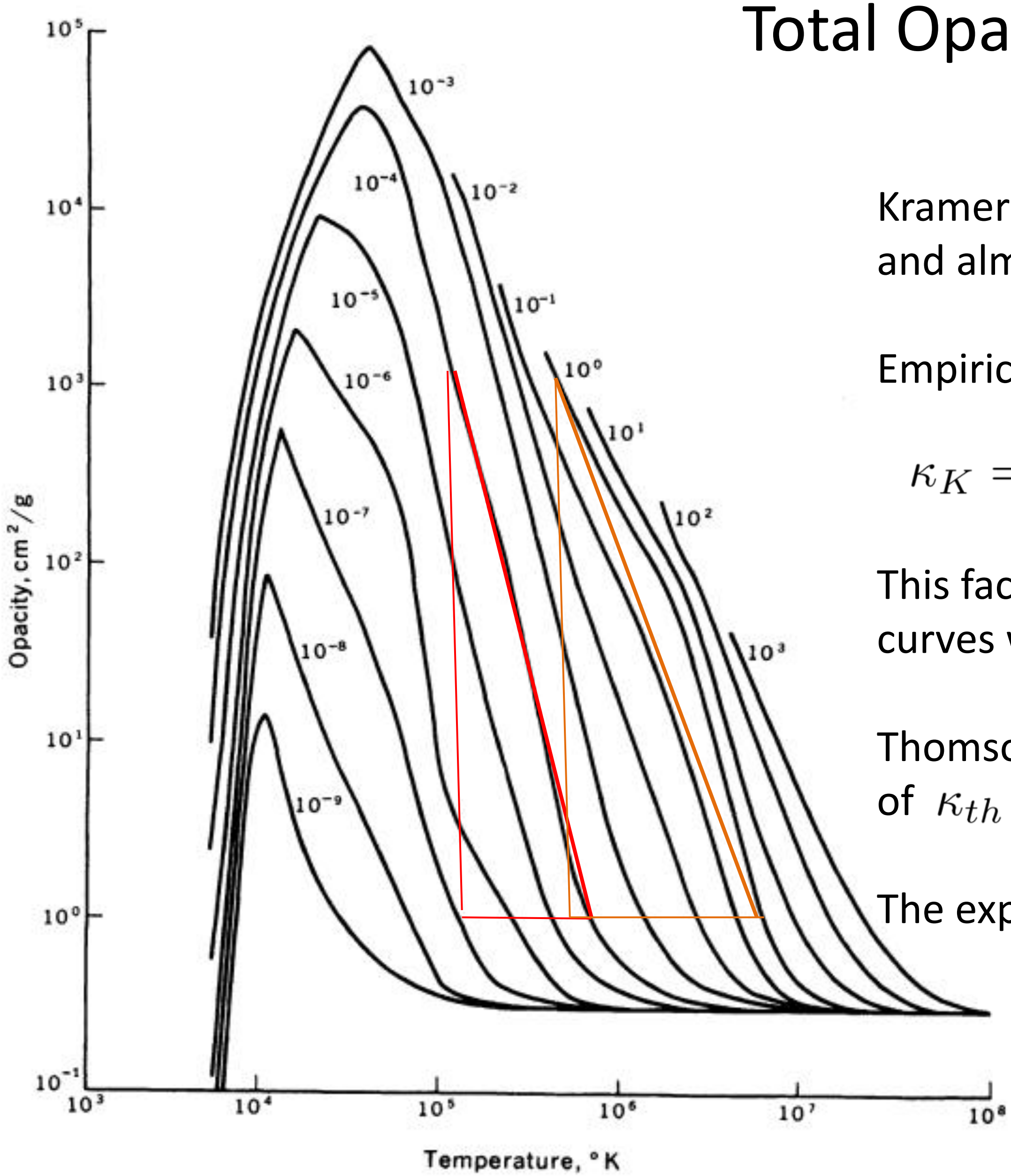


Fig. 3-16 The total opacity of material of solar composition as a function of temperature. Each curve is labeled by the value of the density. The range of values

Kramer's Opacity: Varies (crudely) as $T^{-3.5}$ and almost as $\propto \rho$

Empirical relation given as:

$$\kappa_K = 1.2 \times 10^{24} \frac{(1+X_H)}{2} \left(\frac{0.1}{\rho}\right)^k \frac{\rho}{T^{3.5}}$$

This factor of 2 is required to get agreement for curves with $\rho \geq 0.1 \text{ g cm}^{-3}$

Thomson's Opacity: Constant, and has a value of $\kappa_{th} = 0.40/\mu_e \text{ cm}^2\text{g}^{-1}$

The exponent in Kramer's Opacity is also: $k = 0.3$

We need to simplify the Kramer's opacity so that it is "averaged" over all of the stars we are considering in the Mass-Luminosity relationship.

The first step: get a formula that expresses temperature in terms of density. This will give us a Kramer's formula that is now only a function of density.

Recall, from page 3, we found (and this is also in Lecture 2/3 pages 36, 37) that:

$$\Rightarrow P_{tot} = \left(\frac{3}{a}\frac{1-\beta}{\beta^4}\right)^{1/3}\left(\frac{kN_A}{\mu}\right)^{4/3}\rho^{4/3}$$

$$\beta^{-1}P_g = \beta^{-1}\frac{N_A\rho}{\mu}kT$$

Get his from squaring and rearranging the stellar mass formula on page 6.

$$M_* = 18.0\frac{\sqrt{1-\beta}}{\mu^2\beta^2}M_\odot$$

You have all the ingredients here to now relate temperature to density. The final result, after doing the algebra:

$$T = 4.6\times 10^6\mu\beta\left(\frac{M_*}{M_\odot}\right)^{2/3}\rho^{1/3}$$

Use this last result in the Kramer's Opacity formula by replacing : ρ/T^3

$$\kappa_K = 1.2 \times 10^{24} \frac{(1+X_H)}{2} \left(\frac{0.1}{\rho}\right)^k \frac{\rho}{T^{3.5}}$$

$$= 3.89 \frac{(1+X_H)}{2} (\mu\beta)^{-3} \left(\frac{M_\odot}{M_*}\right)^2 \left(\frac{0.1}{\rho}\right)^k \left(\frac{10^7}{T}\right)^{1/2}$$

Now, no two stars are alike, so we have to start doing some reasonable averages of T and ρ.

First thing is to average over the stellar temperature. And remember, the temperature of the ideal gas and photon gas are the same. However, the photon gas pressure depends only on the temperature (not on density), so there seems like a good place to start.

Try a volume average of the radiation temperature

$$\langle aT^4 \rangle = \frac{4\pi \int_0^R aT^4 r^2 dr}{4\pi \int_0^R r^2 dr}$$

From the polytrope formalism, the solution to the Lane-Emden $\phi(r)$ equation gives the run of density as a function of radial coordinate, r. For n=3 polytrope, we had (page 29 of Lec. 2/3) $\rho = \rho_c \phi^3$. On previous slide, we had $T \propto \rho^{1/3} \Rightarrow T = T_c \phi$.

The above integrals can be done numerically (Mathematica), using numerical $\phi(r)$ Result is:

$$T_{av} = \langle T^4 \rangle^{1/4} = 0.322 T_c$$

We will need T_c

We now have an "average" temperature (weighted over volume) in terms of central temperature. Next, we need the density that corresponds with this T_{av}

Two slides ago (slide 9) we had the following result:

$$T = 4.6 \times 10^6 \mu\beta \left(\frac{M_*}{M_\odot} \right)^{2/3} \rho^{1/3}$$

$$\Rightarrow \frac{T_{av}}{T_c} = 0.322 = \left(\frac{\rho_{av}}{\rho_c} \right)^{1/3}$$

Need to eliminate this

And from the Polytrope formalism in, you(!) should have found the following result:

$$\frac{\bar{\rho}}{\rho_c} = -3\left(\frac{1}{\xi}\frac{d\phi}{d\xi}\right)_{\xi_*}$$

Where, $\bar{\rho} \equiv \frac{M_*}{4\pi R_*^3/3}$

$$= \frac{1}{54.1825}$$

From the table on page 4.

Finally, for the Sun (a Main Sequence Star), $\bar{\rho} = 1.404 \text{ g cm}^{-3}$

$$\rho_c = 54.1825\bar{\rho} = 54.1825 \times 1.404 = 76.1 \text{ g cm}^{-3}$$

$$\rho_{av} = 0.322^3 \times \rho_c = 0.322^3 \times 76.1 = 2.54 \text{ g cm}^{-3}$$

What have we got now: $T_{av} = 0.322T_c$ and $\rho_{av} = 2.54 \text{ g cm}^{-3}$

And we need to complete: $\kappa_K = 3.89\frac{(1+X_H)}{2}(\mu\beta)^{-3}\left(\frac{M_\odot}{M_*}\right)^2\left(\frac{0.1}{\rho_{av}}\right)^k\left(\frac{10^7}{T_{av}}\right)^{1/2}$

We still need the central temperature, and then we are DONE!

The central temperature:

$$T_c = 4.6 \times 10^6 \mu\beta \left(\frac{M_*}{M_\odot}\right)^{2/3} \rho_c^{1/3}$$

From the last slide, we had: $\dfrac{\bar{\rho}}{\rho_c} = -3 \left(\dfrac{1}{\xi}\dfrac{d\phi}{d\xi}\right)_{\xi_*} = \dfrac{1}{54.1825}$

And: $\bar{\rho} = 1.404 \text{ g cm}^{-3}$

For a Solar-type star: $X = 0.71 \quad Y = 0.27 \quad Z = 0.02$ and $\beta \approx 1$

$$\mu = \left[\frac{X}{1.008} n_H + \frac{Y}{4.004} n_{He} + Z \left\langle \frac{n_z}{A_z} \right\rangle\right]^{-1} = 0.613$$

Assuming fully ionized

Collecting all the numbers, we finally have: $T_c = 11.9 \times 10^6 \text{ K}$

$$\Rightarrow T_{av} = 0.322 T_c = 3.85 \times 10^6 \text{ K}$$

Finally, Kramer's Opacity becomes simplified to:

$$\kappa_K = 3.89\frac{(1+X_H)}{2}(\mu\beta)^{-3}\left(\frac{M_\odot}{M_*}\right)^2\left(\frac{0.1}{\rho_{av}}\right)^k\left(\frac{10^7}{T_{av}}\right)^{1/2} \approx 1.19(1+X_H)(\mu\beta)^{-3}\left(\frac{M_\odot}{M_*}\right)^2$$

Total Opacity: $\kappa = \kappa_{th} + \kappa_K$ where $\kappa_{th} = \frac{0.40}{\mu_e}$

And, for fully ionized material: $\frac{1}{\mu_e} = \frac{(1+X)}{2}$ And X = 0.71 for Solar.

The total Opacity is now: $\kappa = 0.342 + 2.03(\mu\beta)^{-3}\left(\frac{M_\odot}{M_*}\right)^2$

And we had for Luminosity: $L_* = 1.35 \times 10^{35}\frac{(\mu\beta)^4\left(\frac{M_*}{M_\odot}\right)^3}{\kappa}\ \mathrm{erg/s}$

Calling $m = \frac{M_*}{M_\odot}$ we finally have the function for Luminosity:

$$L_* = 1.35 \times 10^{35} \frac{(\mu\beta)^7 m^5}{0.342 m^2 (\mu\beta)^3 + 2.03} \text{ erg/s}$$

Or, using $L_\odot = 3.84 \times 10^{33} \text{ erg s}^{-1}$

$$\boxed{\frac{L_*}{L_\odot} = 103 \frac{(\mu\beta)^7 m^5}{m^2 (\mu\beta)^3 + 5.94} \text{ erg/s}}$$

With μ , as before, given by its Solar value:

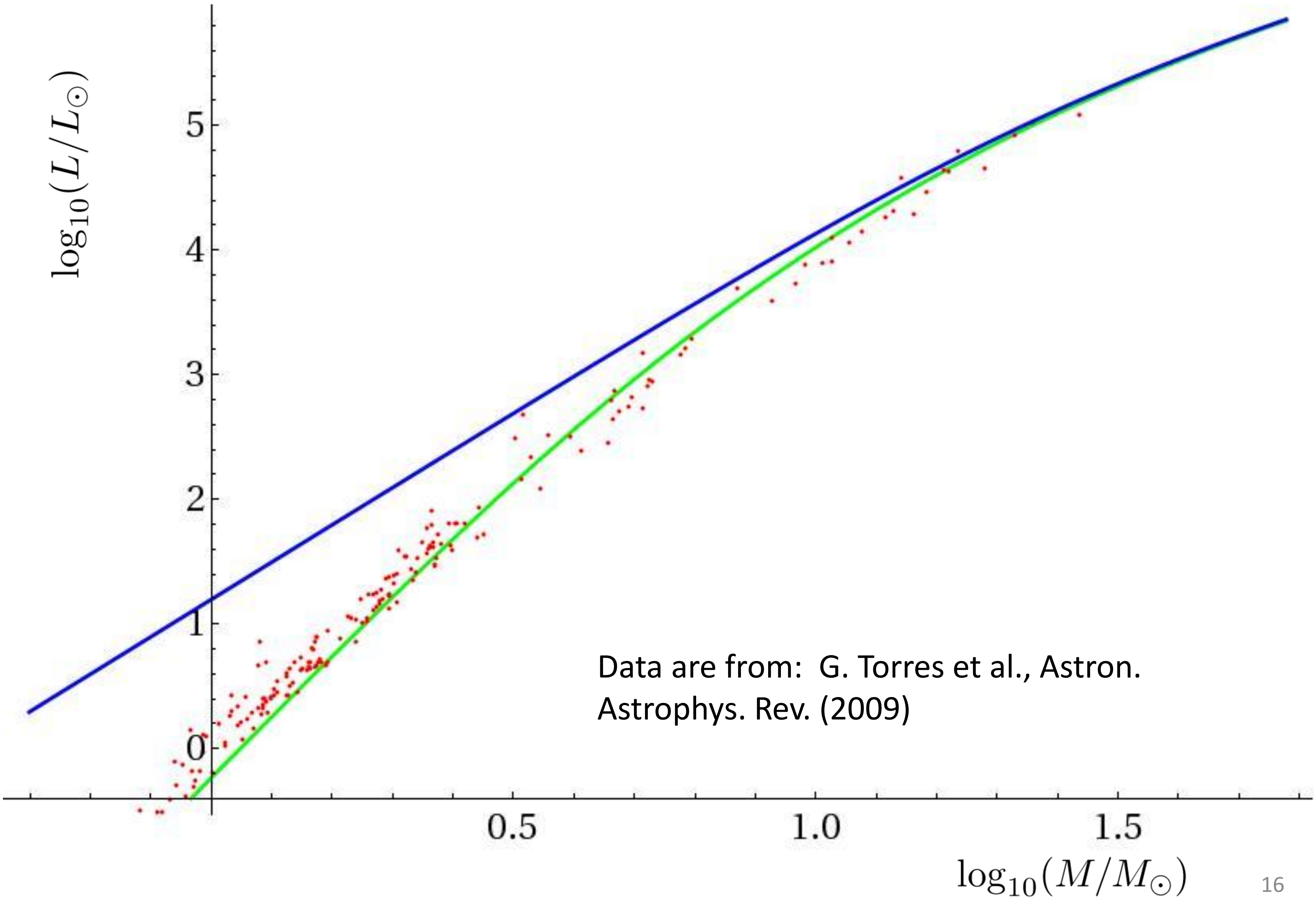
$$\mu = \left[\frac{X}{1.008} n_H + \frac{Y}{4.004} n_{He} + Z \left\langle \frac{n_z}{A_z} \right\rangle \right]^{-1} = 0.613$$

Assuming fully ionized

The function above is parametric in β . We work the function by choosing a value for m, and then solving Eddington's Quartic equation for β, to evaluate the RHS.

What does it look like when plotted against REAL Main Sequence data??

Mass-Luminosity: Main Sequence



Data are from: G. Torres et al., Astron. Astrophys. Rev. (2009)