Nuclear Astrophysics

Lecture 8

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Alternative Rate Formula(s)

We had from last lecture:

$$r_{12} = \left(\frac{2}{\mu}\right)^{1/2} \frac{N_1 N_2}{1+\delta_{12}} \tau^{-3/2} \Delta S_0 \exp(-3E_{\text{eff}}/\tau)$$

With:

$$E_{\text{eff}} = \left(\frac{b\tau}{2}\right)^{2/3} = 1.22(Z_1^2 Z_2^2 \mu T_6^2)^{1/3} \ \ \text{keV}$$

And:

$$\Delta = \frac{4}{\sqrt{3}}(E_{\text{eff}}\ \tau)^{1/2} = 0.75(Z_1^2 Z_2^2 \mu T_6^5)^{1/6}$$

And (pg 16, Lecture 6):

$$b = 2\pi \frac{Z_1 Z_2 e^2}{\hbar} \left(\frac{\mu}{2}\right)^{1/2} = 31.27 Z_1 Z_2 \mu^{1/2} \ \ \text{keV}^{1/2}$$

And we have approximated the S-factor, at the astrophysical energies, as a simple constant function S_0 (units of keV barn).

We have (again) $r_{12} = \left(\frac{2}{\mu}\right)^{1/2} \frac{N_1 N_2}{1+\delta_{12}} \tau^{-3/2} \Delta S_0 \exp(-3E_{\text{eff}}/\tau)$

Set the argument of the exponential to be $\chi = 3E_{\text{eff}}/\tau$ (dimensionless)

From the definitions of $\Delta,\ E_{\text{eff}},\ \text{and}\ b$ on the previous slide, you can work out that:

$$\frac{\Delta}{\tau^{3/2}} = \frac{4}{9\sqrt{3}} \left(\frac{2}{b}\right) \chi^2$$

$$\Rightarrow r_{12} = \frac{4}{9\sqrt{3}} \left(\frac{2}{b}\right) \left(\frac{2}{\mu}\right)^{1/2} \frac{N_1 N_2}{1+\delta_{12}} \chi^2 S_0 \exp(-\chi)$$

$$= 7.20 \times 10^{-19} \frac{N_1 N_2}{1+\delta_{12}} \frac{S_0}{\mu Z_1 Z_2} \chi^2 e^{-\chi}$$

A Power Law expression for the Rate

$$r_{12} = \left(\frac{2}{\mu}\right)^{1/2} \frac{N_1 N_2}{1+\delta_{12}} \tau^{-3/2} \Delta S_0 \exp(-3E_{\text{eff}}/\tau)$$

First, an algebraic step of substituting E_{eff} into the equation for Δ

$$\Delta = \frac{4}{\sqrt{3}} \left(\frac{b}{2}\right)^{1/3} \tau^{5/6}$$

We substitute this, along with E_{eff} into r_{12} above. After some algebra:

$$r_{12} = \underbrace{\left(\frac{32}{3\mu}\right)^{1/2} \left(\frac{b}{2}\right)^{1/3} \frac{N_1 N_2}{1+\delta_{12}} S_0}_{A} \tau^{-2/3} \exp\left[-3\left(\frac{b}{2}\right)^{2/3} \tau^{-1/3}\right]$$

$$= A\tau^{-2/3} \exp\left[-3\left(\frac{b}{2}\right)^{2/3} \tau^{-1/3}\right]$$

Let's pull out the physics of this complicated formula by trying to write it as something simpler, like a power law function.

First, equate the last expression for the rate to something like a power law:

$$r_{12} = r_{12}(\tau_0)\left(\frac{\tau}{\tau_0}\right)^n = A\tau^{-2/3}\exp\left[-3\left(\frac{b}{2}\right)^{2/3}\tau^{-1/3}\right]$$

Take the natural logarithm of both sides:

$$\ln r_{12}(\tau_0) + n(\ln\tau - \ln\tau_0) = \ln A - \frac{2}{3}\ln\tau - 3\left(\frac{b}{2}\right)^{2/3}\tau^{-1/3}$$

Differentiate both sides wrt $\ln\tau$: $\left(\frac{\partial}{\partial\ln\tau}\right)$

$$n = -\frac{2}{3} + \left(\frac{b}{2}\right)^{2/3}\tau^{-1/3}$$

$$= \left(-2 + 3\frac{E_{\mathrm{eff}}}{\tau}\right)/3$$

Numerically, $$3\frac{E_{\rm eff}}{\tau} = 42.487\left(\frac{Z_1^2 Z_2^2 \mu}{T_6}\right)^{1/3}$$

Summarizing then:

Choose a value of temperature τ_0 to evaluate the rate at:

$$r_{12}(\tau_0) = \left(\frac{32}{3\mu}\right)^{1/2}\left(\frac{b}{2}\right)^{1/3}\frac{N_1 N_2}{1+\delta_{12}} S_0 \tau_0^{-2/3} \exp\left[-3\left(\frac{b}{2}\right)^{2/3}\tau_0^{-1/3}\right]$$

Once we have this value for the rate at some temperature **you** have chosen, to get the rate at any other temperature, just make the trivial calculation:

$$r_{12}(T) = r_{12}(T_0)\left(\frac{T}{T_0}\right)^{(-2+3\frac{E_{\rm eff}}{\tau})/3}$$

$$r_{12}(T_0) = 7.20\times 10^{-19}\frac{N_1 N_2}{1+\delta_{12}}\frac{S_0}{\mu Z_1 Z_2}\chi^2(T_0)e^{-\chi(T_0)}$$

Finally, recall $N_i = \rho N_A \frac{X_i}{A_i}$

$$r_{12} = \frac{2.62 \times 10^{29} \rho^2}{1 + \delta_{12}} \frac{X_1 X_2}{A_1 A_2} \frac{S_0}{\mu Z_1 Z_2} \chi^2 e^{-\chi}$$

Summary of Reaction Rate

$$r_{12} = \frac{2.62 \times 10^{29} \rho^2}{1 + \delta_{12}} \frac{X_1 X_2}{A_1 A_2} \frac{S_0}{\mu Z_1 Z_2} \chi^2 e^{-\chi}$$

$$r_{12}(T) = r_{12}(T_0) \left(\frac{T}{T_0} \right)^{(-2+3\frac{E_{\rm eff}}{\tau})/3}$$

$$\chi = 3E_{\rm eff}/\tau$$

$$E_{\rm eff} = \left(\frac{b\tau}{2} \right)^{2/3} = 1.22(Z_1^2 Z_2^2 \mu T_6^2)^{1/3} \quad \text{keV}$$

$$\tau = kT = 0.086 T_6$$

Examples:

$$E_{\rm eff} = \left(\frac{b\tau}{2}\right)^{2/3} = 1.22(Z_1^2 Z_2^2 \mu T_6^2)^{1/3} \ \ \text{keV}$$

The core of our Sun has a central temperature of about 15 million degrees kelvin.

Reaction	Z_1	Z_2	mu	T_6	E_{eff}	n
p + p	1	1	0.5	15	5.88	3.89
p + ^{14}N	1	7	0.933	15	26.53	19.90
^{4}He + ^{12}C	2	6	3	15	56.09	42.81
^{16}O + ^{16}O	8	8	8	15	237.44	183.40

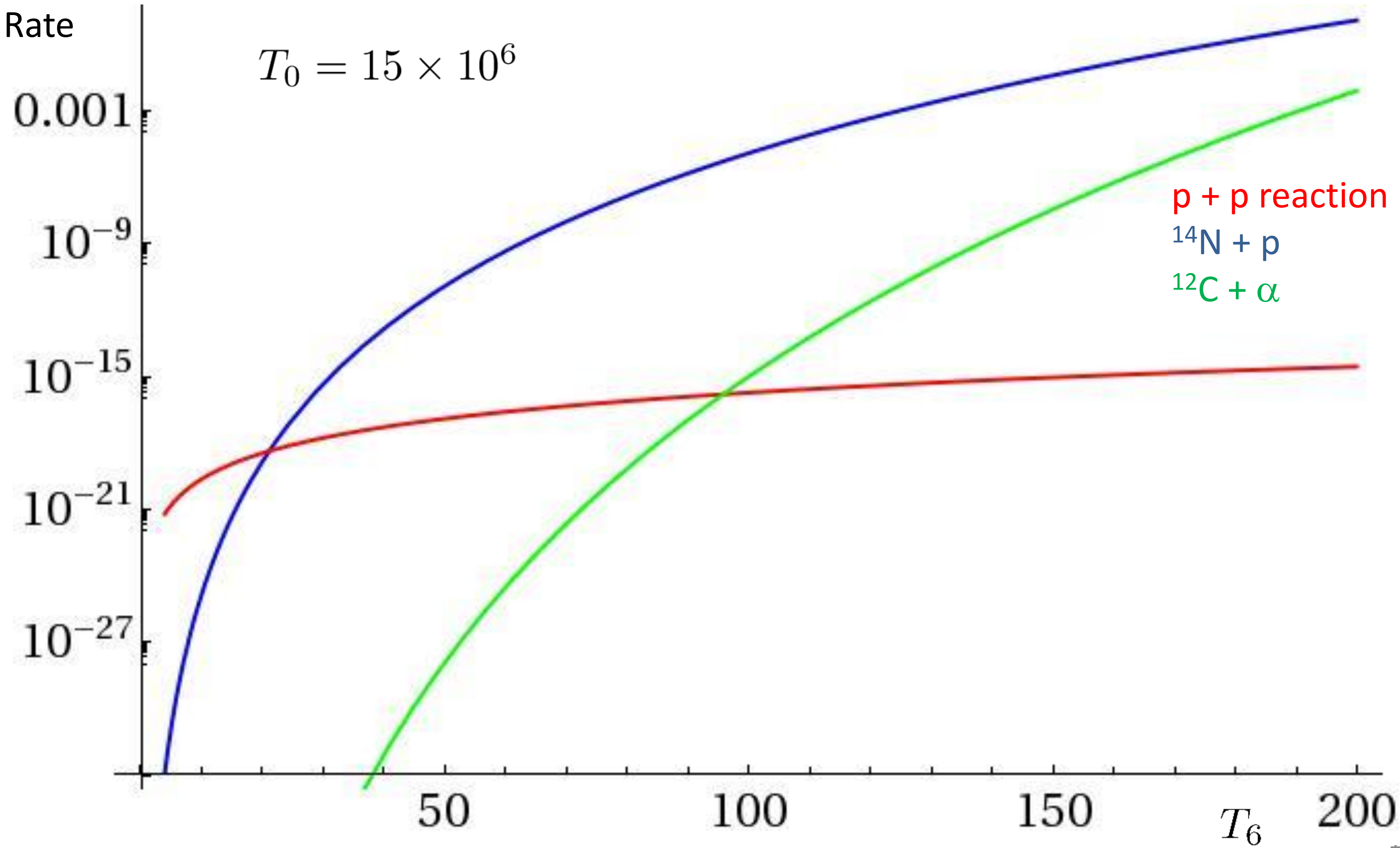
Reaction rate becomes more sensitive to temperature as we “burn” heavier nuclei together.

Such a large sensitivity to the temperature suggests structural changes in star must occur at some point for certain reactions.

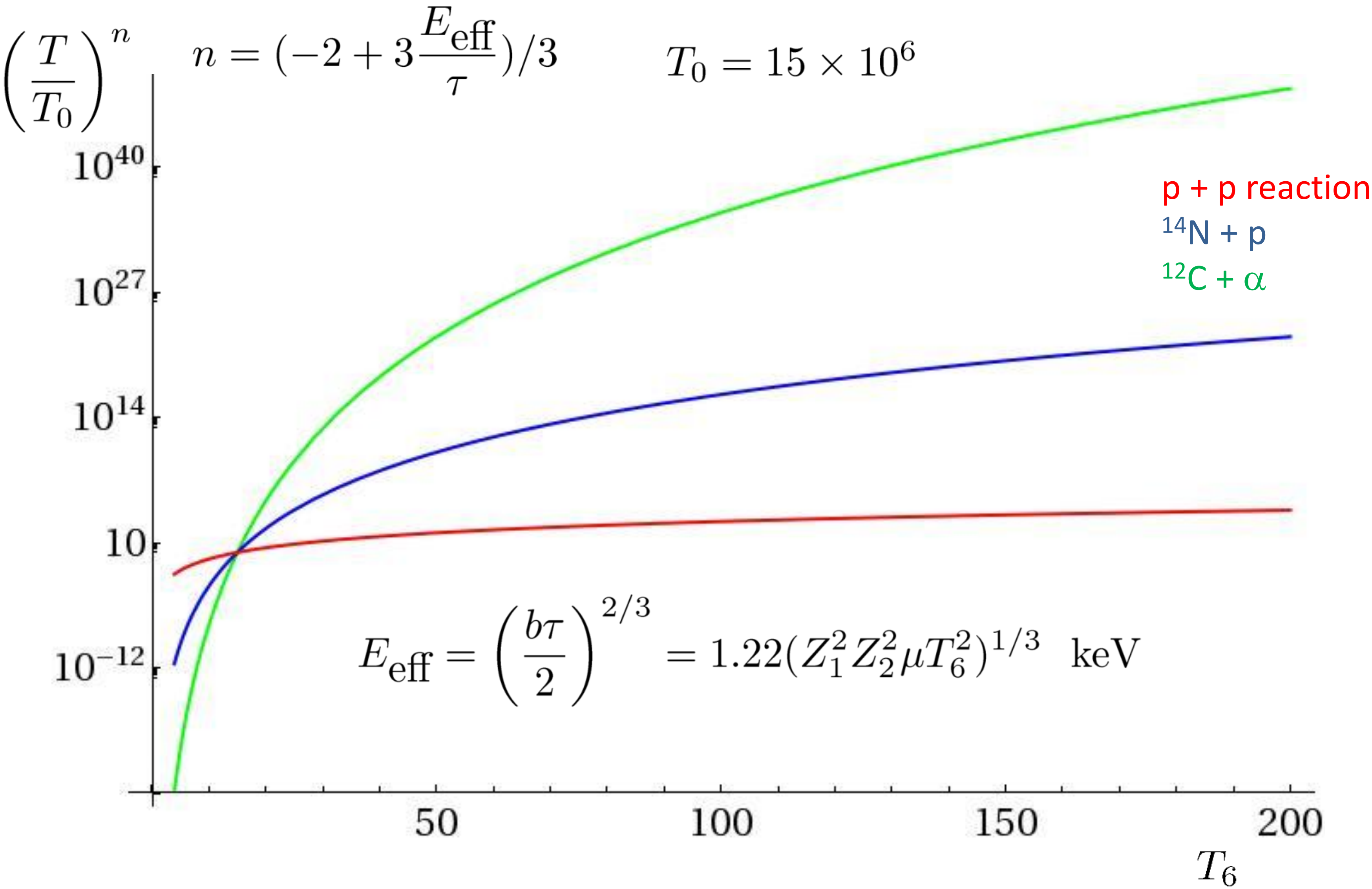
Full Reaction Rates: Comparison

Solar abundances, density = 100 g cm^{-3}:

$$X_H = 0.71 \qquad X_{^{12}C} = 3 \times 10^{-3} \qquad X_{^{14}N} = 1.1 \times 10^{-3}$$



Reaction Rate: Temperature Dependence



Energy Generation & Standard Model

Let us consider a representation for the nuclear rate of energy production (in units of energy per unit time, per unit mass) to be given by the following mathematical form:

$$\epsilon \propto \rho^{u-1} T^s$$

Then we can, quite generally, also write:

$$\frac{\epsilon}{\epsilon_c} = \left(\frac{\rho}{\rho_c}\right)^{u-1} \left(\frac{T}{T_c}\right)^s$$

For a Polytrope model, once the solution ϕ to Lane-Emden equation is known, then we have the following results (for n = 3 Polytrope):

$$\frac{\rho(r)}{\rho_c} = \phi^3(r) \qquad \frac{T(r)}{T_c} = \phi(r) \qquad \frac{P(r)}{P_c} = \phi^4(r)$$

Using the equations on previous slide, we can now write the scaled nuclear energy generation rate in the very compact and simple form:

$$\frac{\epsilon_r}{\epsilon_c} = \phi_r^{3u+s-3}$$

The luminosity of the star (after settling into equilibrium) comes from the nuclear furnace in the core. Let's try to write the luminosity in the following form:

$$L = M_* \bar{\epsilon} \qquad (*)$$

Where we use the mass-averaged energy generation rate:

$$\bar{\epsilon} = \frac{1}{M_*} \int_0^{M_*} \epsilon_r dM_r$$

We use an averaged energy rate because it is clear that not all mass in the star is contributing to the nuclear energy rate: only the core of the star is contributing to the energy generation rate. So, we average over the mass of the star in the hope that the averaged rate times total mass (equation *) gives a reasonable result.

We must now consider the integral:

$$\bar{\epsilon} = \frac{1}{M_*} \int_0^{M_*} \epsilon_r dM_r$$

Our differential mass element is, in the **scaled** radial coordinate $r = a\xi$ (refer to Lecture 2):

$$dM_r = 4\pi a^3 \rho \xi^2 d\xi = 4\pi a^3 \rho_c \phi^3 \xi^2 d\xi$$

And, from previous page, $\epsilon_r = \epsilon_c \phi_r^{3u+s-3}$

Collecting everything into the integral:

$$\bar{\epsilon} = \frac{4\pi a^3 \rho_c \epsilon_c}{M_*} \int_0^{\xi_*} \phi^{3u+s} \xi^2 d\xi$$

In Lecture 3 we had:

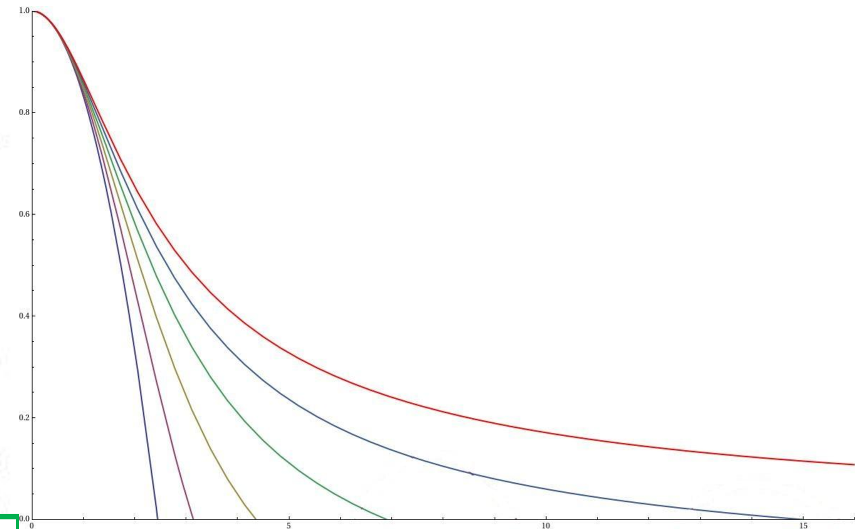
$$M_* = -4\pi \rho_c a^3 \left(\xi^2 \frac{d\phi}{d\xi} \right)_{\xi_*}$$

Finally, then, we have that the mass-averaged nuclear energy rate is given by:

$$\bar{\epsilon} = \frac{\epsilon_c}{\left(-\xi^2 \frac{d\phi}{d\xi}\right)_{\xi_*}} \int_0^{\xi_*} \phi^{3u+s} \xi^2 d\xi \ \approx \ \frac{\epsilon_c}{2} \int_0^{\xi_*} \phi^{3u+s} \xi^2 d\xi$$

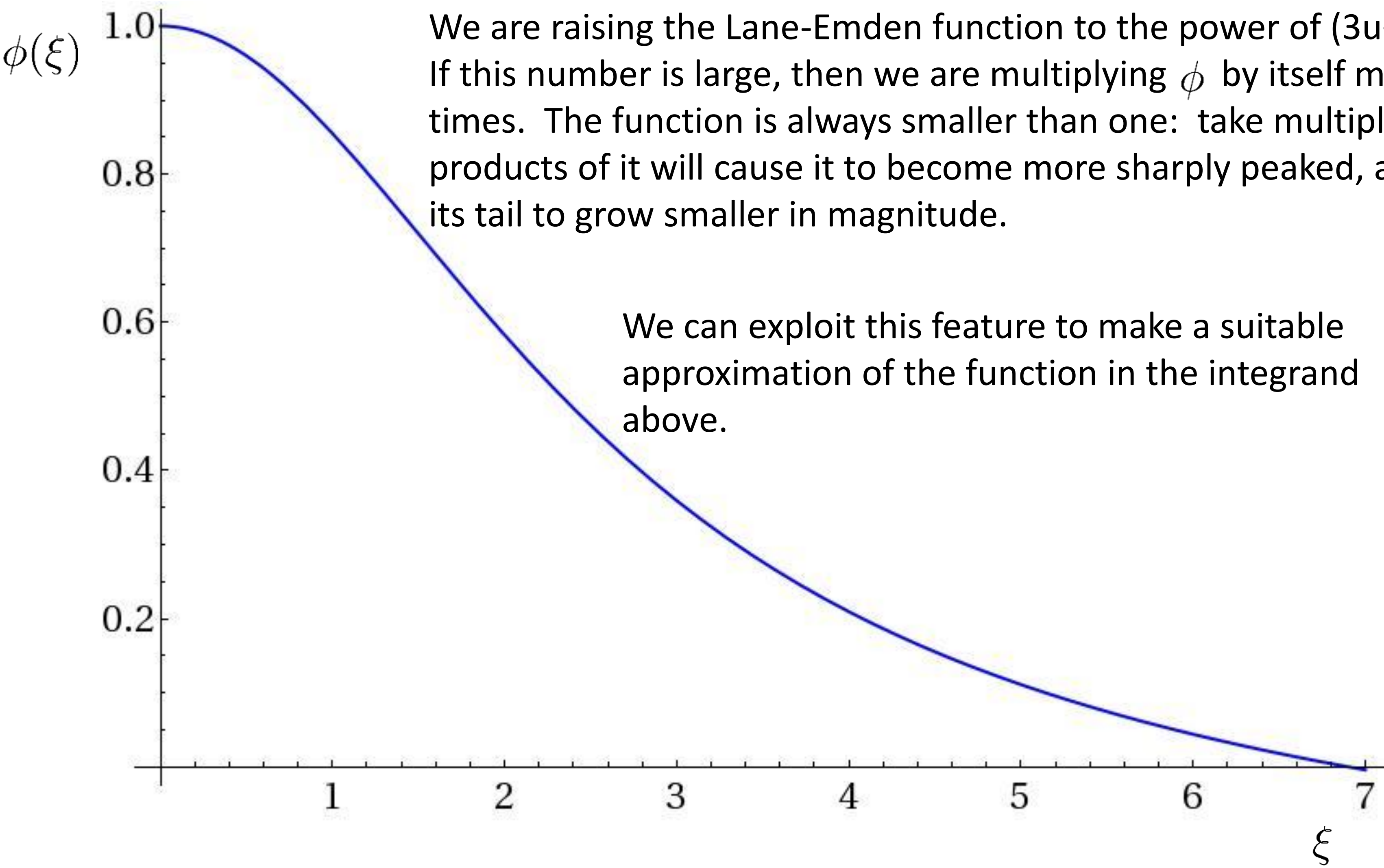
Table 2-5 Constants of the Lane-Emden functions†

n	ξ_1	$-\xi_1^2 \left(\frac{d\phi}{d\xi}\right)_{\xi=\xi_1}$	$\frac{\rho_c}{\bar{\rho}}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6,189.47
4.9	169.47	1.7355	934,800
5.0	∞	1.73205	∞



Lane-Emden Function for n = 3

The integral we must consider is $\bar{\epsilon} = \frac{\epsilon_c}{2}\int_0^{\xi_*} \phi^{3u+s}\xi^2 d\xi$



We are raising the Lane-Emden function to the power of (3u+s). If this number is large, then we are multiplying ϕ by itself many times. The function is always smaller than one: take multiple products of it will cause it to become more sharply peaked, and its tail to grow smaller in magnitude.

We can exploit this feature to make a suitable approximation of the function in the integrand above.

The power series expansion of ϕ_n, for the first few terms looks like this:

$$\phi = 1 - \frac{\xi^2}{6} + n\frac{\xi^4}{120} - \ldots \overset{n=3}{=} 1 - \frac{\xi^2}{6} + \frac{\xi^4}{40} - \ldots \approx \exp\left(-\frac{\xi^2}{6}\right)$$

Let's use the exponential approximation in the integral for $\bar{\epsilon}$

$$\bar{\epsilon} = \frac{\epsilon_c}{2}\int_0^{\xi_*} \phi^{3u+s}\xi^2 d\xi \approx \frac{\epsilon_c}{2}\int_0^{\infty} \exp\left[-(3u+s)\frac{\xi^2}{6}\right]\xi^2 d\xi$$

$$= \frac{\epsilon_c}{2}\left(\frac{6}{3u+s}\right)^{3/2}\int_0^{\infty}\exp\left[-v^2\right]v^2 dv$$

$$= \frac{\epsilon_c}{2}\left(\frac{6}{3u+s}\right)^{3/2}\frac{\pi^{1/2}}{4}$$

Mass-averaged nuclear energy rate:

$$\Rightarrow \bar{\epsilon} = \frac{\epsilon_c}{2}\frac{\sqrt{27\pi/2}}{(3u+s)^{3/2}} \approx \frac{3.26}{(3u+s)^{3/2}}\epsilon_c$$

A Polytrope Sun

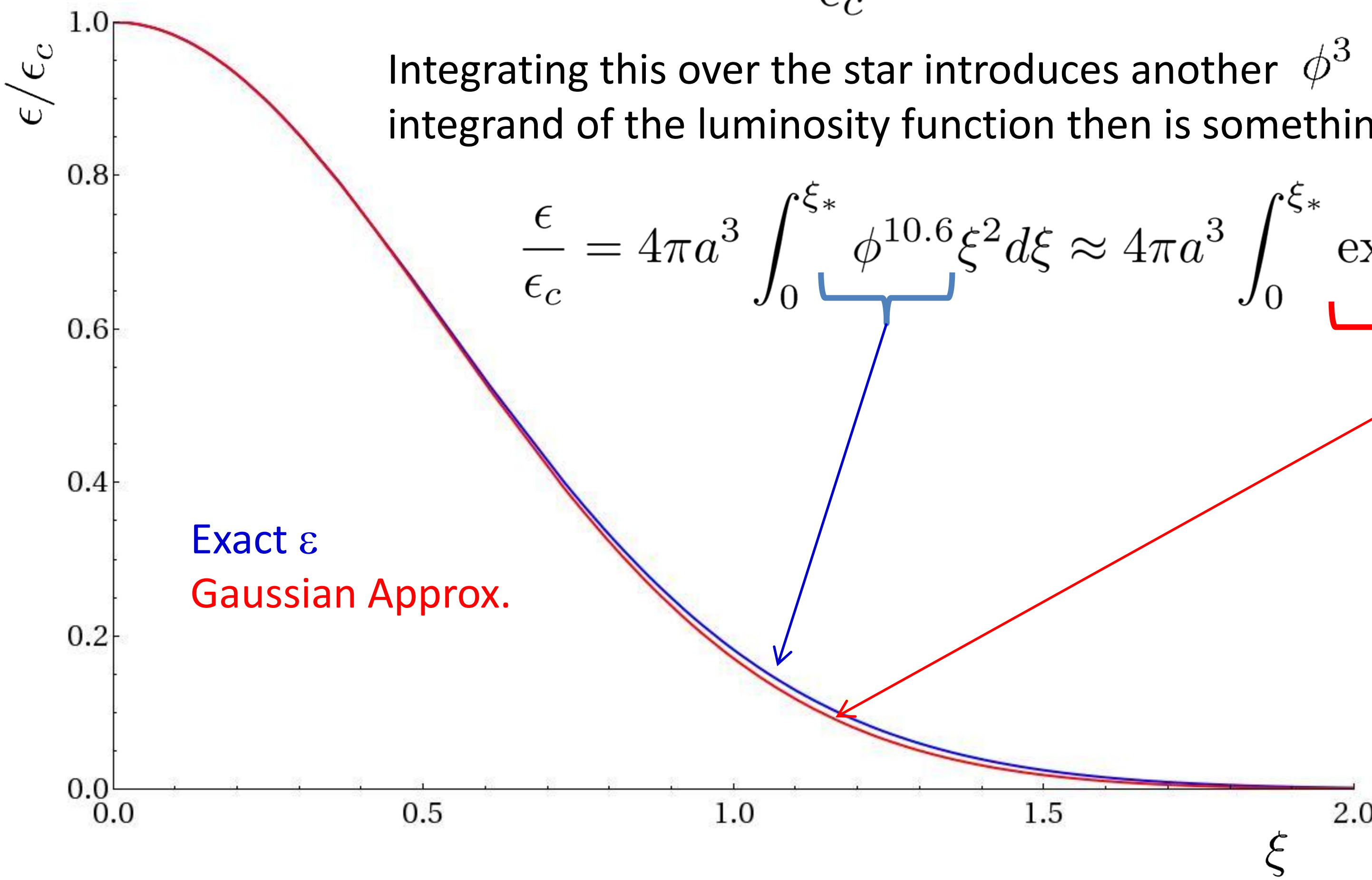
We had the general energy generation rate expressed as: $\frac{\epsilon}{\epsilon_c} = \left(\frac{\rho}{\rho_c}\right)^{u-1}\left(\frac{T}{T_c}\right)^{s}$

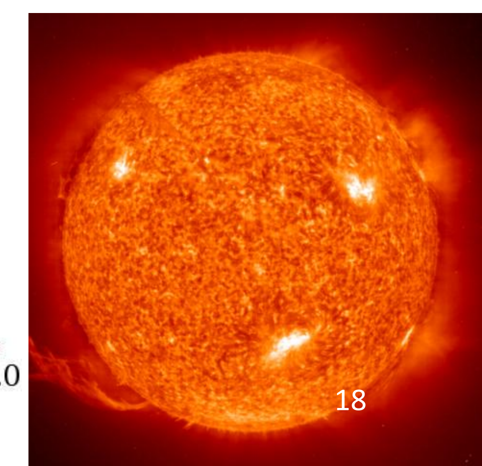
For the proton-proton chain in the Sun, $u = 2$ and $s = 4.6$ (at $T_0 = 10^7\ \mathrm{K}$).
Therefore, we have, in terms of ϕ that:

$$\frac{\epsilon}{\epsilon_c} = \phi^{3+4.6} = \phi^{7.6}$$

Integrating this over the star introduces another ϕ^3 from dM_r. The integrand of the luminosity function then is something like:

$$\frac{\epsilon}{\epsilon_c} = 4\pi a^3 \int_0^{\xi_*} \phi^{10.6}\xi^2 d\xi \approx 4\pi a^3 \int_0^{\xi_*} \exp\left[-\frac{10.6\xi^2}{6}\right]\xi^2 d\xi$$





Structure of Polytrope Sun

Let us define the core to be that volume in which 95% of the nuclear energy is generated.

We find that this occurs around $\xi = 1.6$ (page 20). Recall the stellar radius in the ξ variable is $\xi_* = 6.89$.

So the core occupies about 23% of the total stellar radius.

So the fractional volume occupied by the core is $(r_{core}/R_*)^3 \approx 1.2\%$

Mass occupied by the core: From plot on next page, it is about 33% of stellar mass.

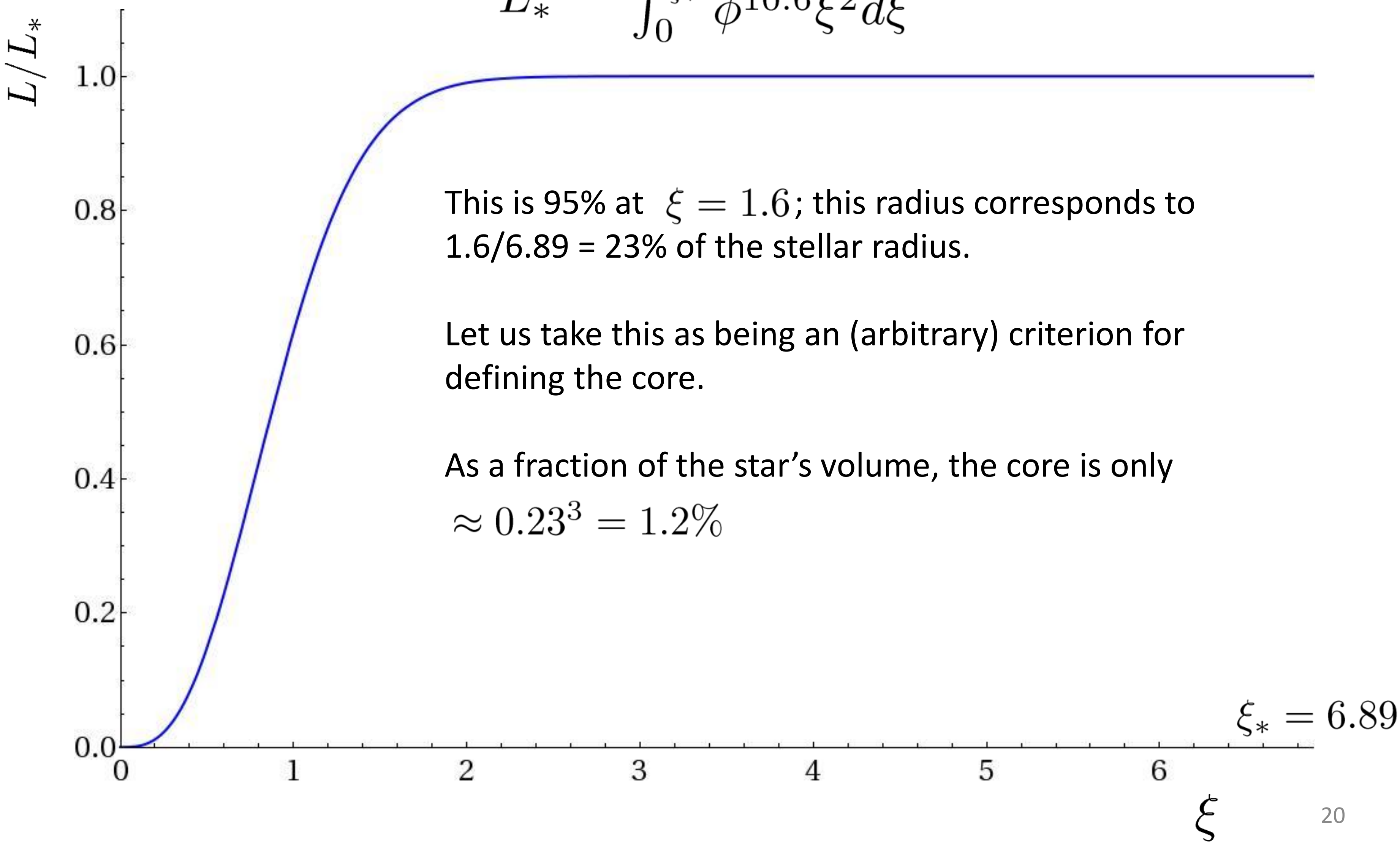
Question for you:

This sun is 1 solar mass, and 1 solar radius in size. Take $\epsilon_c = 0.068 \text{ erg g}^{-1}\text{s}^{-1}$

Use these numbers and the result on page 17 to determine the mass of hydrogen per second the Sun burns. Also use: u =2, and s = 4.6

Integrated Luminosity

$$\frac{L}{L_*} = \frac{\int_0^{\xi} \phi^{10.6} \xi^2 d\xi}{\int_0^{\xi_*} \phi^{10.6} \xi^2 d\xi}$$

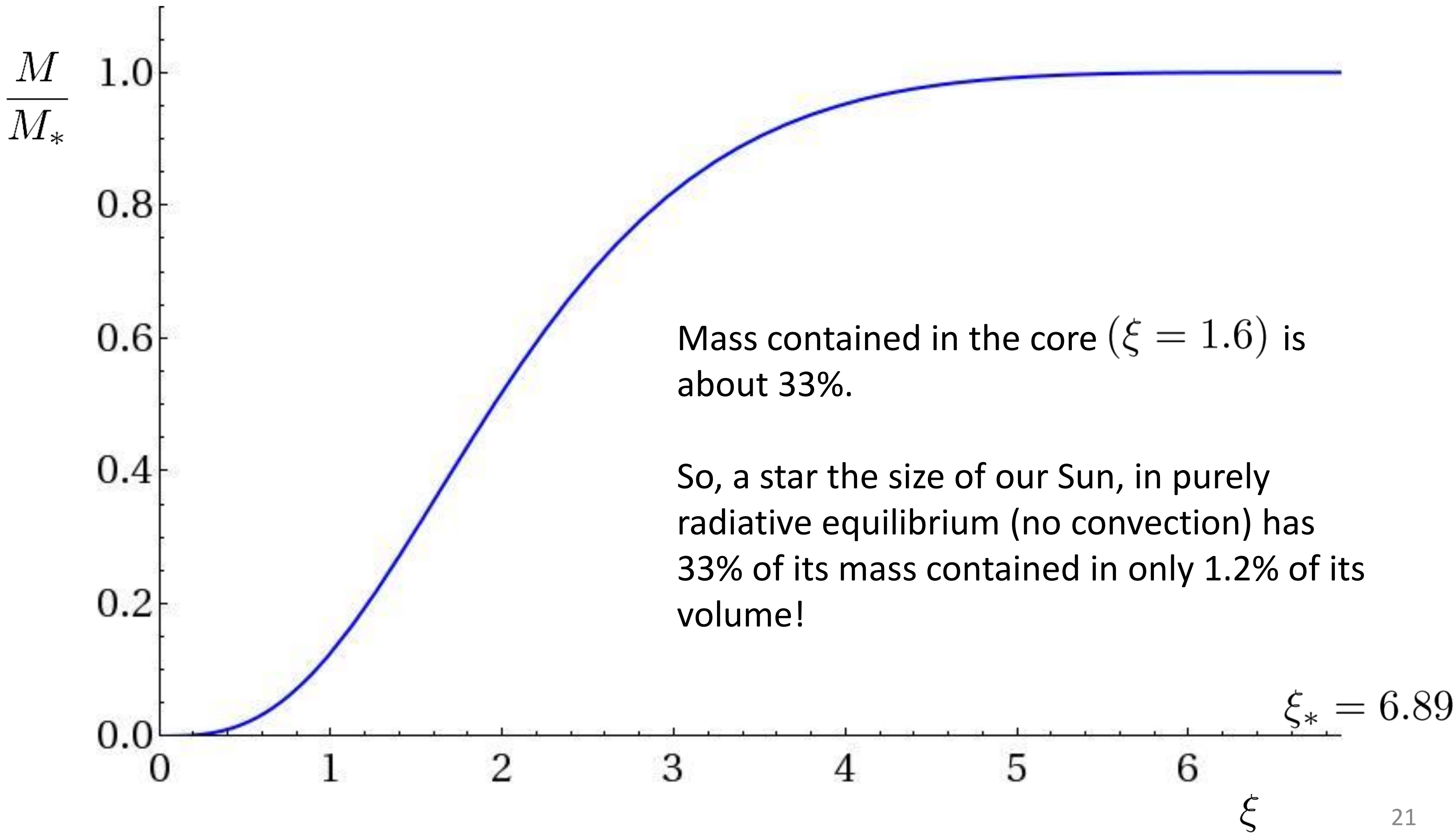


This is 95% at $\xi = 1.6$; this radius corresponds to 1.6/6.89 = 23% of the stellar radius.

Let us take this as being an (arbitrary) criterion for defining the core.

As a fraction of the star's volume, the core is only $\approx 0.23^3 = 1.2\%$

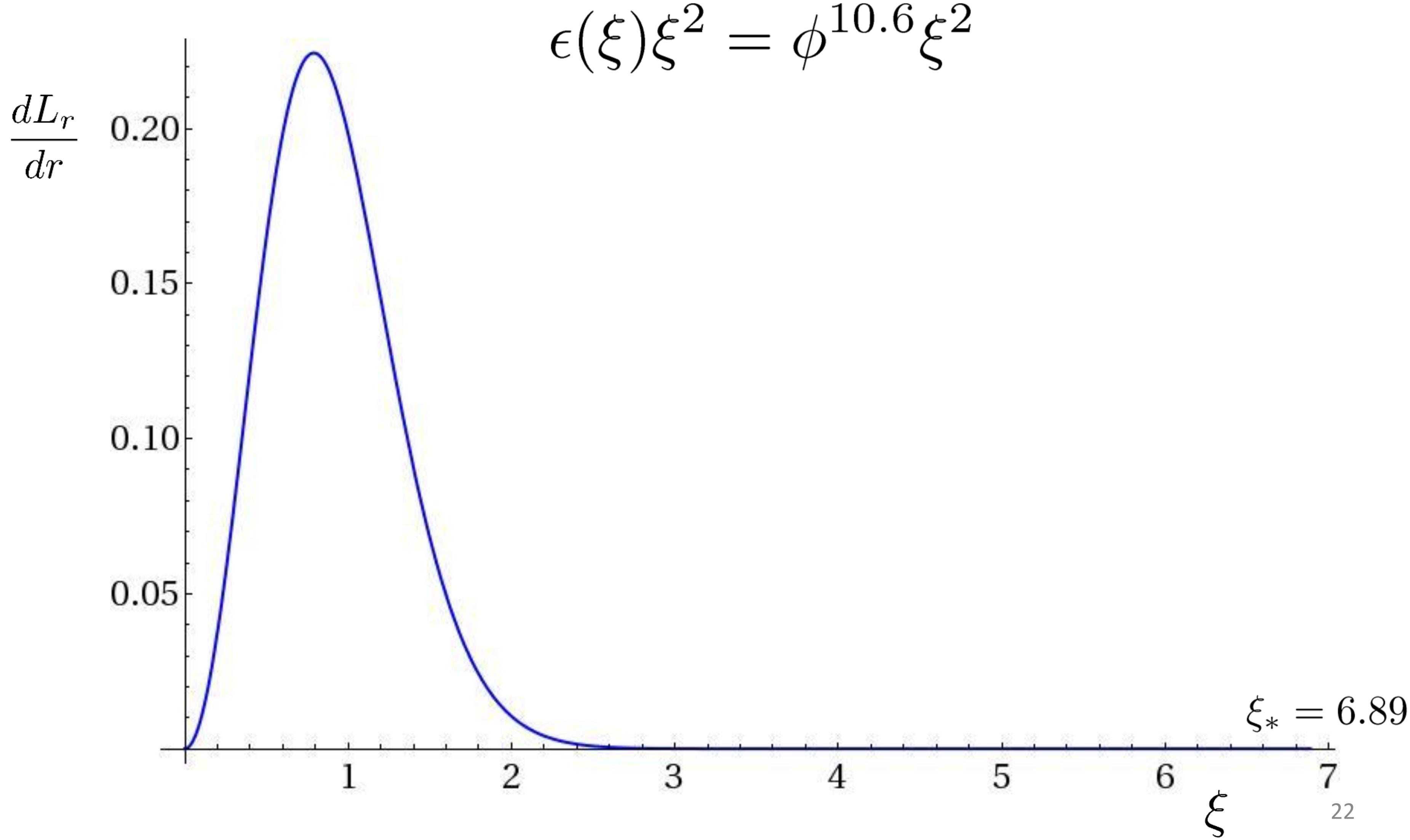
Mass Interior to Radius ξ



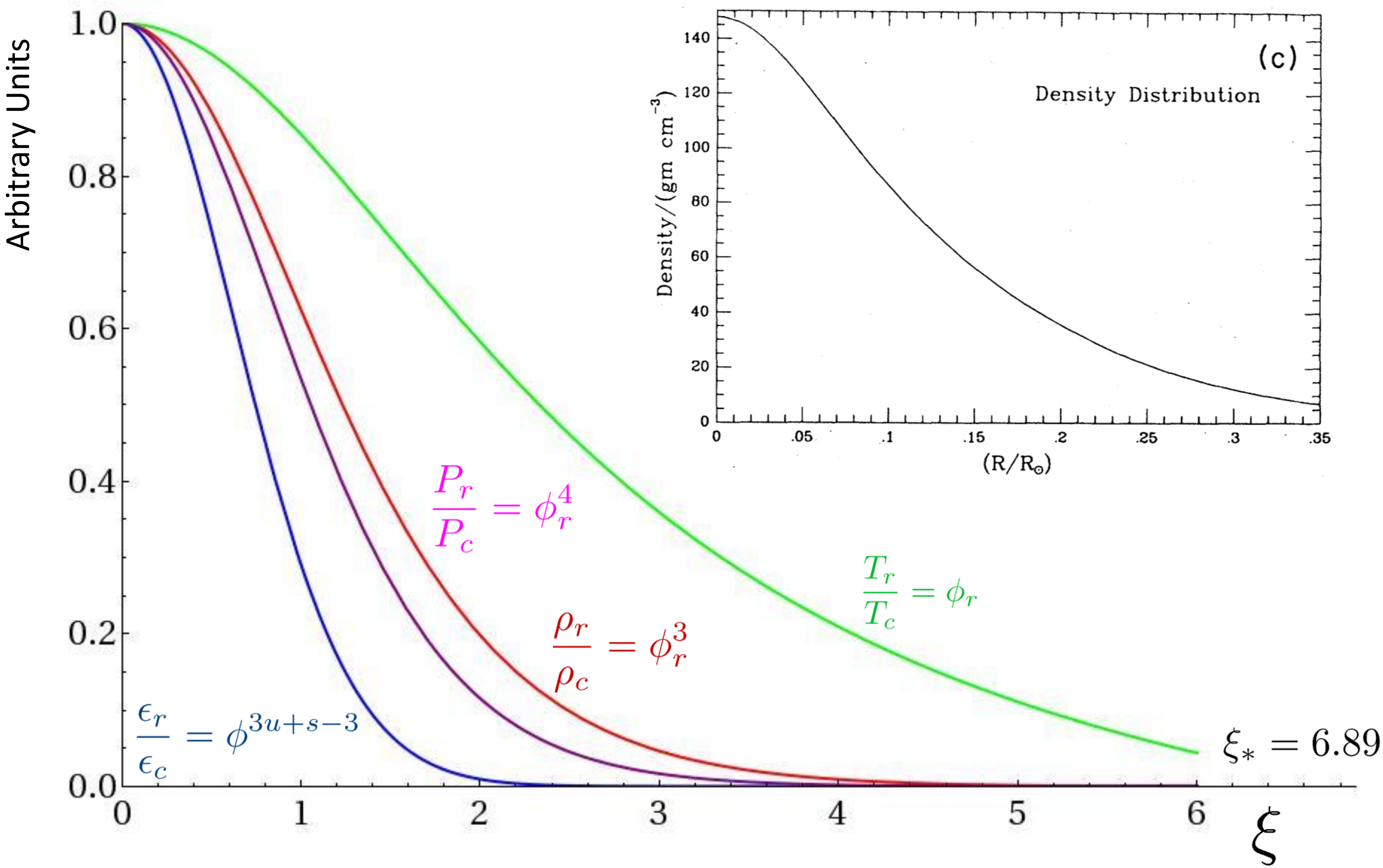
Mass contained in the core $(\xi = 1.6)$ is about 33%.

So, a star the size of our Sun, in purely radiative equilibrium (no convection) has 33% of its mass contained in only 1.2% of its volume!

Derivative of Luminosity



Standard Solar Model: Radial Run of Temperature, Density and Nuclear Energy Rate



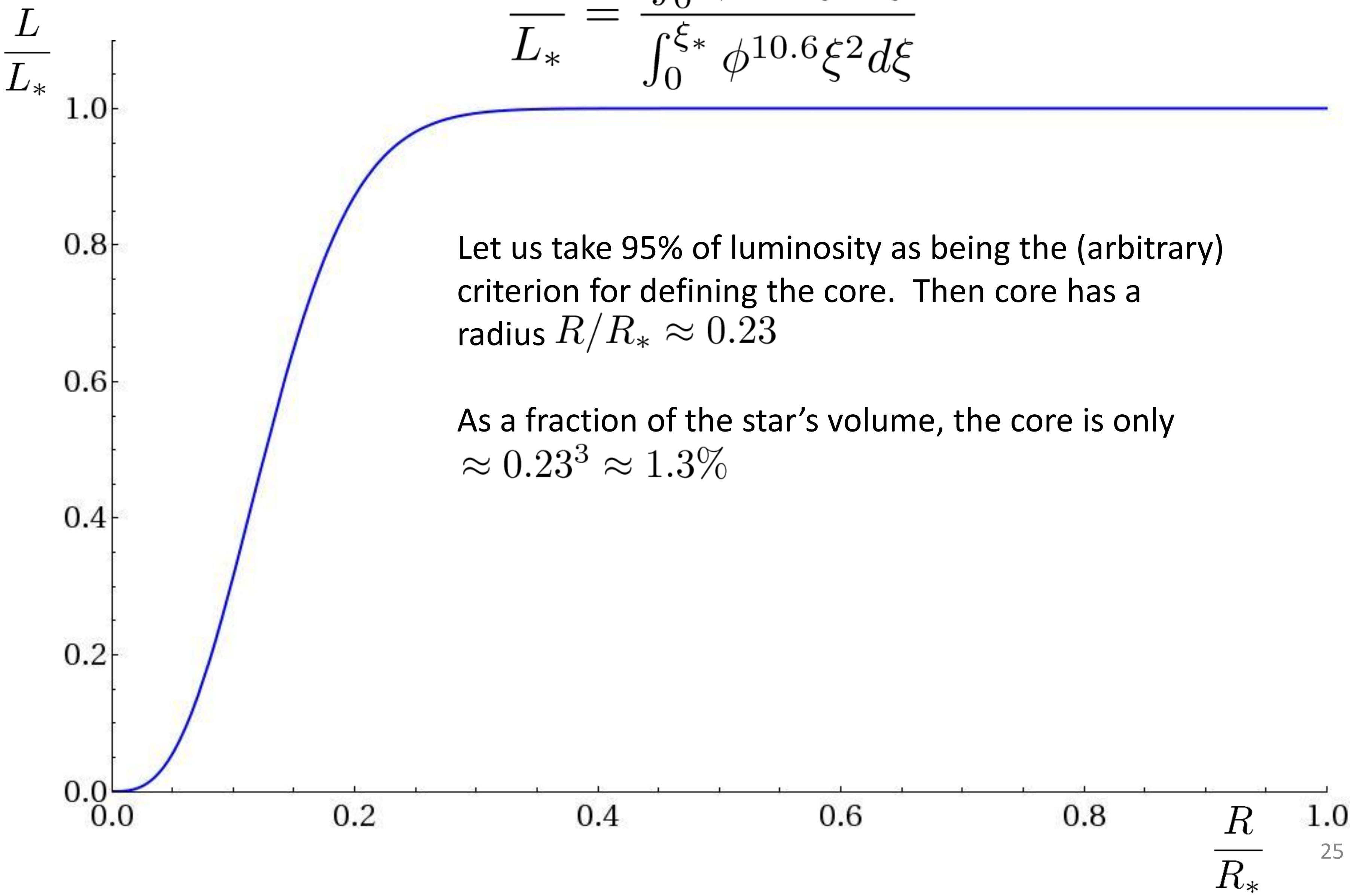
Remember, for n = 3: $\xi_* = 6.89$

Polytrope model, for polytrope index n = 3

APPENDIX: SCALED MAIN SEQUENCE STRUCTURE PLOTS

Integrated Luminosity

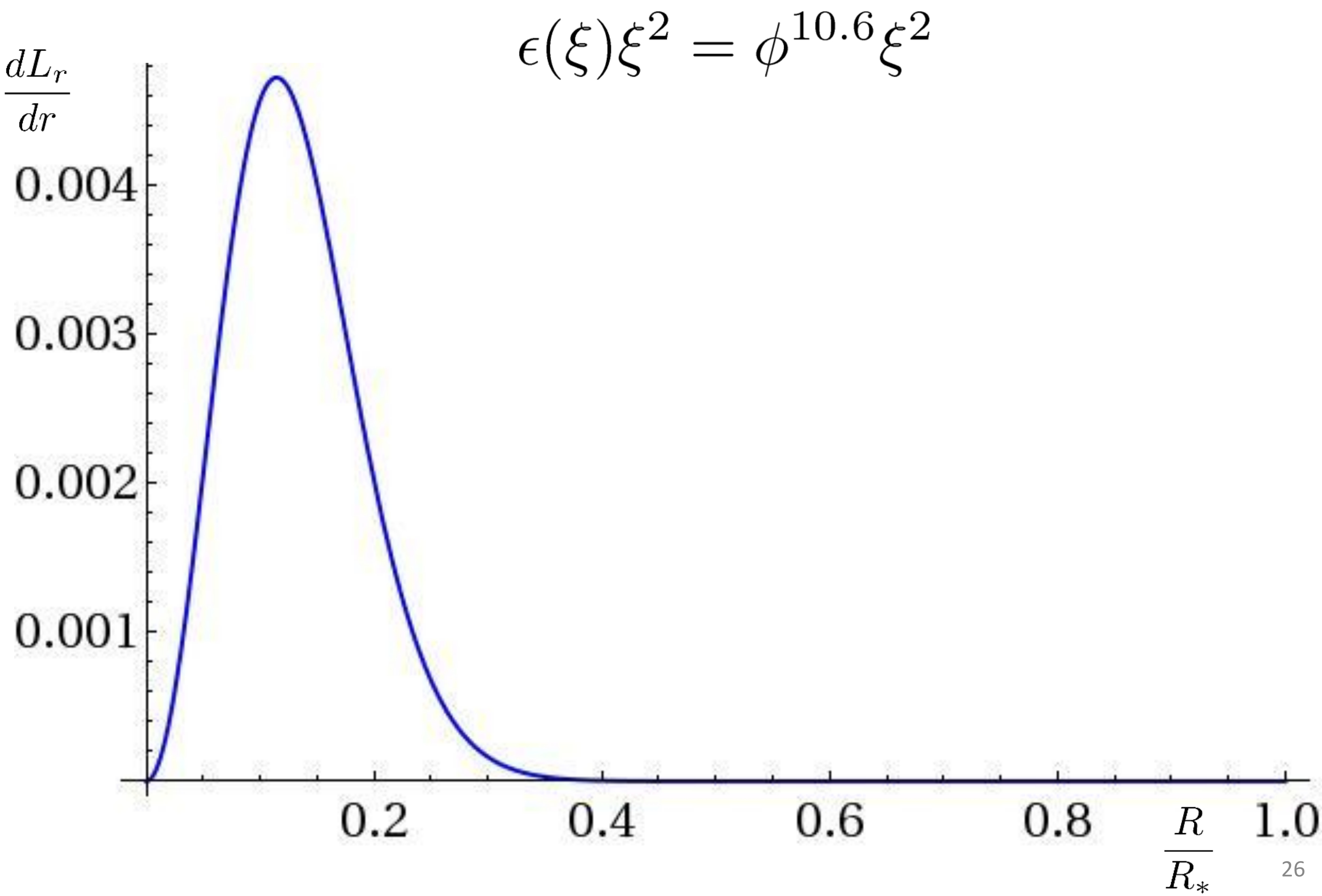
$$\frac{L}{L_*} = \frac{\int_0^{\xi} \phi^{10.6} \xi^2 d\xi}{\int_0^{\xi_*} \phi^{10.6} \xi^2 d\xi}$$



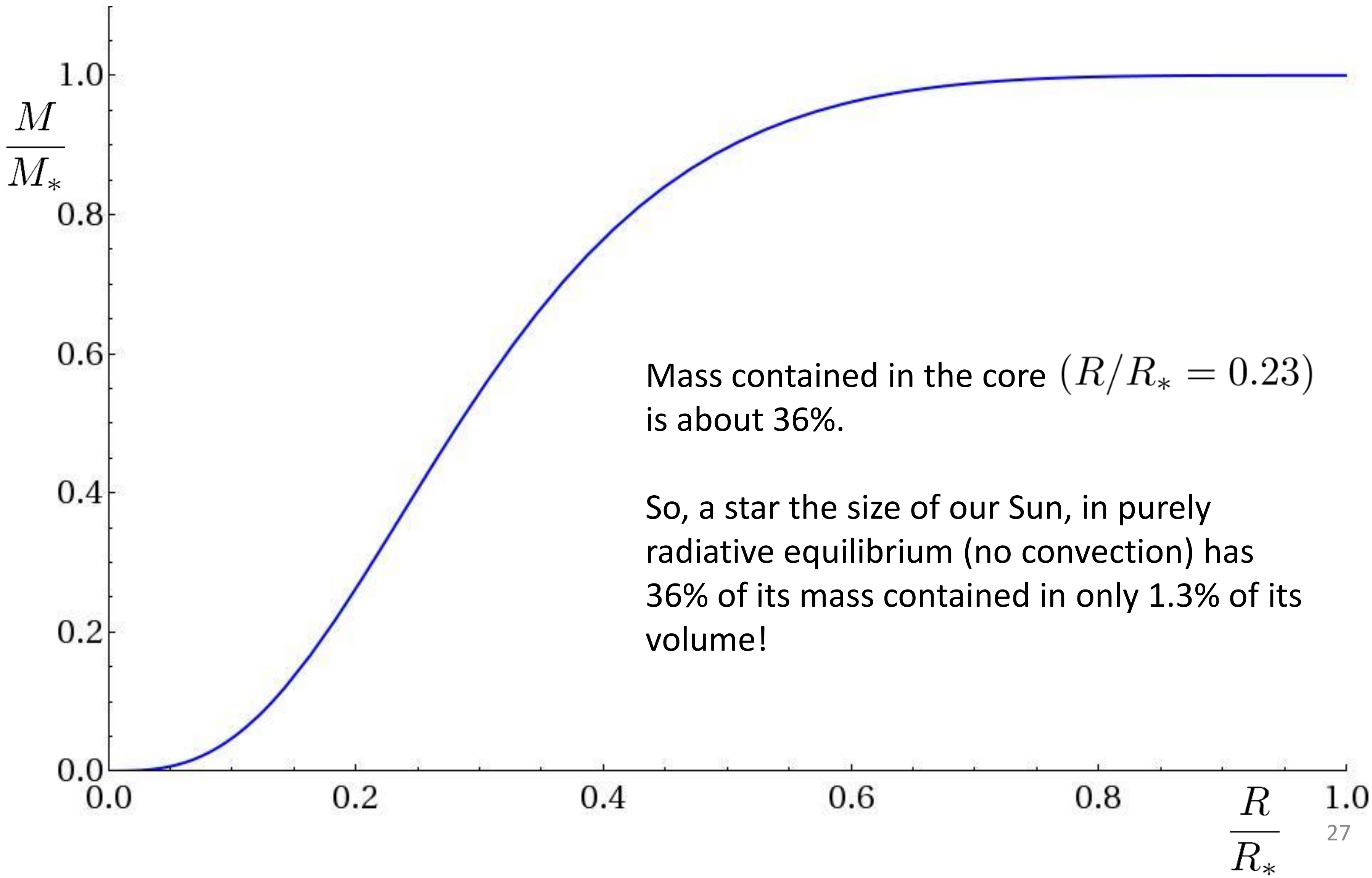
Let us take 95% of luminosity as being the (arbitrary) criterion for defining the core. Then core has a radius $R/R_* \approx 0.23$

As a fraction of the star's volume, the core is only $\approx 0.23^3 \approx 1.3\%$

Derivative of Luminosity



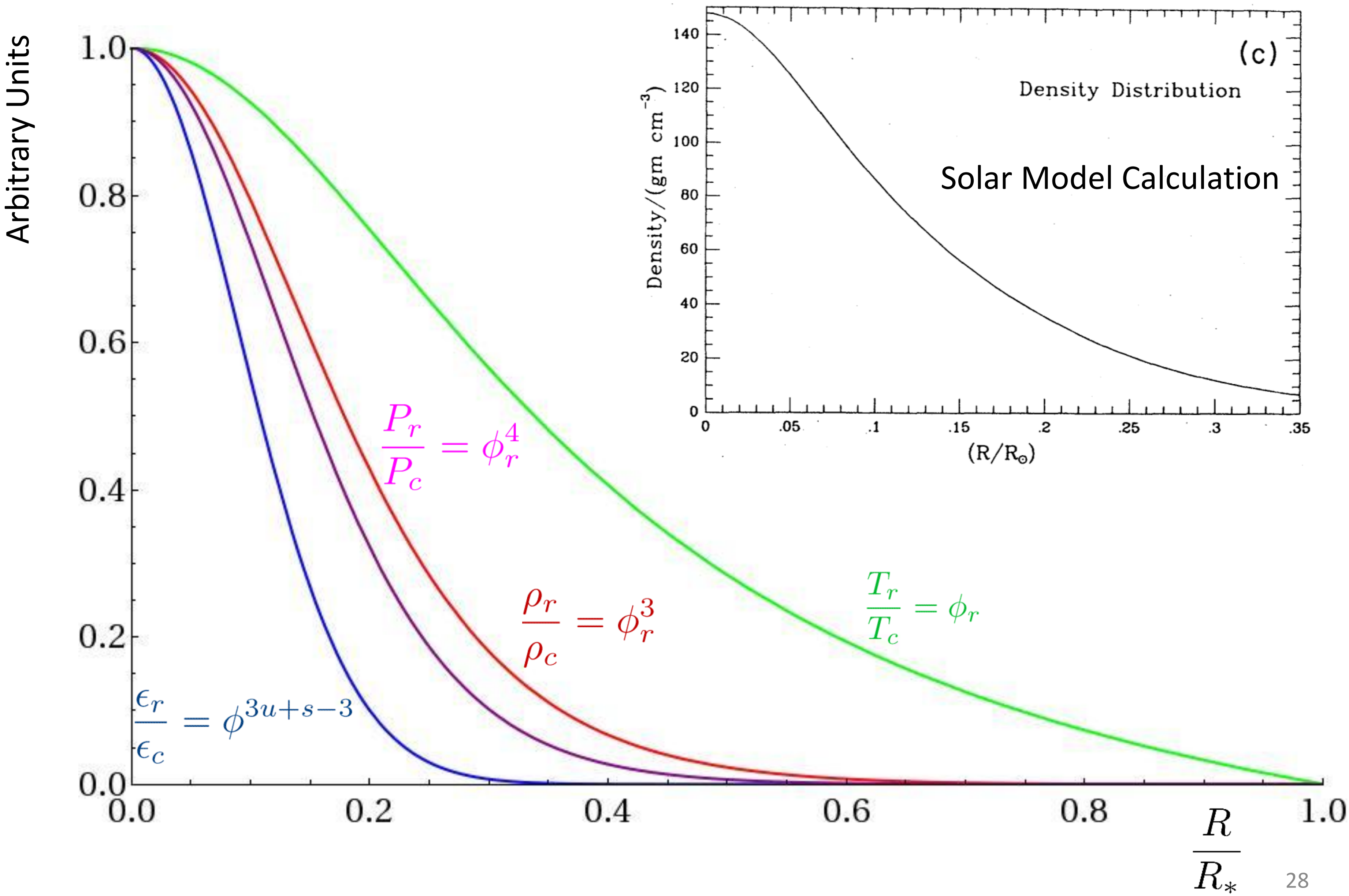
Mass Interior to Radius $\frac{R}{R_*}$



Mass contained in the core $(R/R_* = 0.23)$ is about 36%.

So, a star the size of our Sun, in purely radiative equilibrium (no convection) has 36% of its mass contained in only 1.3% of its volume!

Standard Solar Model: Radial Run of Temperature, Density and Nuclear Energy Rate



Nuclear Generation Rate

Using the **pp-chain** ϵ_c , which is a power law in temperature, with exponent n = 4.6, we end up with the Polytropic energy generation rate expressed on in the integral on the LHS below, where ϕ is the Lane-Emden function of index = 3.

The L-E function, raised to such a high power, can be approximated to an excellent degree by a Gaussian (RHS), which can then be analytically integrated.

$$\frac{\epsilon}{\epsilon_c} = 4\pi a^3 \int_0^{\xi_*} \phi^{10.6}\xi^2 d\xi \approx 4\pi a^3 \int_0^{\xi_*} \exp\left[-\frac{10.6\xi^2}{6}\right]\xi^2 d\xi$$