# Nuclear Astrophysics

Lecture 7

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# Reaction Rate Summary

Reaction rate for charged particles: $1 + 2 \rightarrow 3 + 4$

$$r_{12} = \frac{4\pi N_1 N_2}{1+\delta_{12}} \left(\frac{\mu}{2\pi\tau}\right)^{3/2} \int_0^\infty v^3 \sigma(v) \exp\left(-\frac{\mu v^2}{2\tau}\right) dv$$

$$r_{12} = \left(\frac{8}{\pi\mu}\right)^{1/2} \frac{N_1 N_2}{1+\delta_{12}} \tau^{-3/2} \int_0^\infty E_{12} \sigma_{12}(v) \exp\left(-\frac{E_{12}}{\tau}\right) dE_{12}$$

Reaction rate for photodisintegration (photon in entrance channel): $1 + \gamma \rightarrow 2 + 3$

$$r_{1\gamma} = \frac{8\pi N_1}{h^3 c^2} \int_0^\infty \sigma_{1\gamma}(E_\gamma) \frac{E_\gamma^2}{\exp(E_\gamma/\tau) - 1} dE_\gamma$$

# THE PATH TO CROSS SECTIONS

3-Dimensional SWE, after separation of variables, will produce a radial equation of the following type:

$$\left[-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{\ell(\ell+1)\hbar^2}{2\mu r^2}+V(r)\right]R(r)=ER(r)$$

Make the substitution: $R(r)=\frac{u(r)}{r}$ and show that the above equation becomes

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+\frac{\ell(\ell+1)\hbar^2}{2\mu r^2}+V(r)\right]u(r)=Eu(r)$$

For $r\rightarrow\infty$, and outside the interaction zone, the above equation asymptotically becomes:

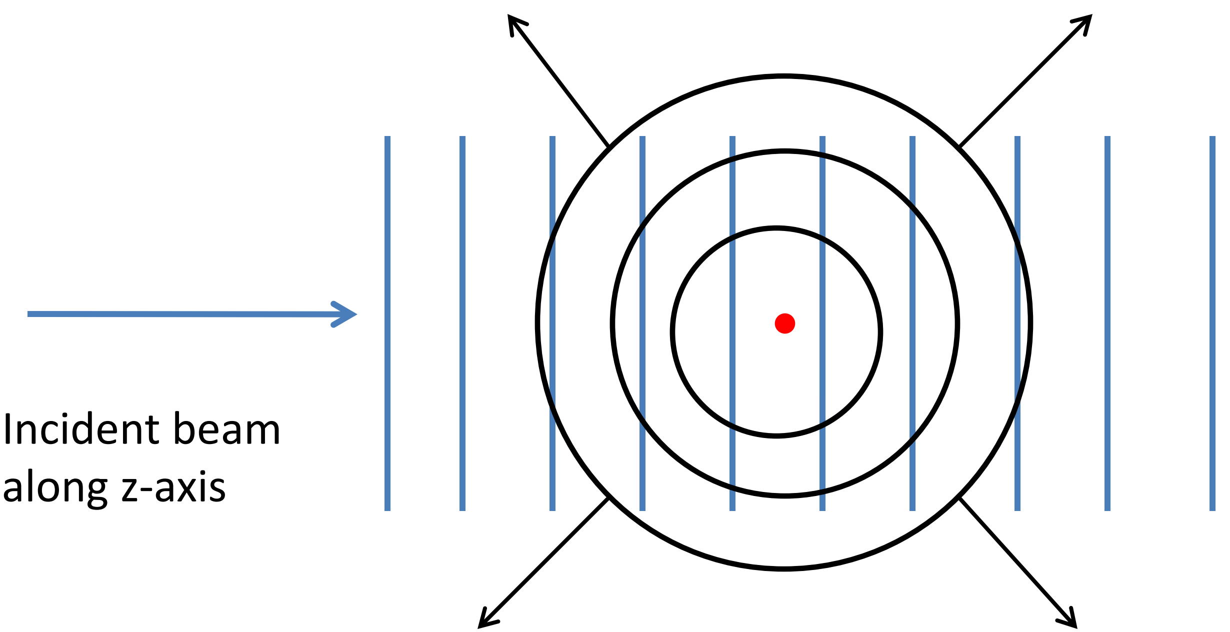
$$-\frac{\hbar^2}{2\mu}\frac{d^2u(r)}{dr^2}=Eu(r)$$

$$-\frac{\hbar^2}{2\mu}\frac{d^2u(r)}{dr^2} = Eu(r)$$ has a solution: $u(r) = \exp \pm ikr$, where $k = \sqrt{\frac{2\mu E}{\hbar^2}}$

For a beam incident from the left, the solution outside the potential zone at large r is just: $u(r) = A \exp ikz$

Outgoing wave function?

Some of the incident beam transmits through the interaction zone, the rest scatters or undergoes a reaction. How to quantify this?



Incident beam along z-axis

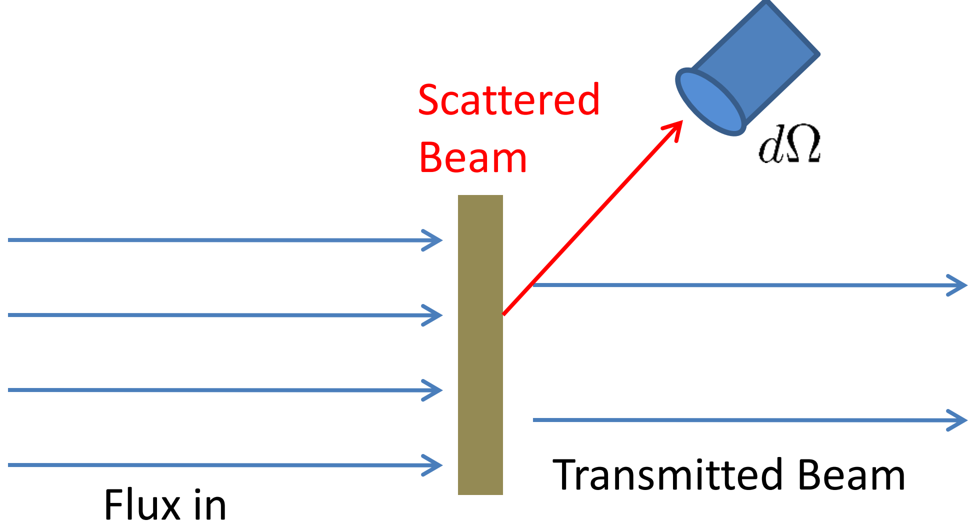
Outgoing wave = Incident wave + Scattering wave

$$\psi = A \exp ikz + Af(\theta, \phi)\frac{\exp ikr}{r} \quad \text{as } r \to \infty$$

Now we need to relate this function and the incident wave function to the reality of what we measure in a nuclear physics experiment.

Consider: $\psi^* \psi$ . This is a (local) probability density for the existence of a particle at the spatial coordinates $(r, \theta, \phi)$ . Its time derivative tells us how the probability of there being a particle at these coordinates evolves in time.

We must find a way to relate this quantity to the reality of an experiment.




Differential cross section is defined as:

$$d\sigma = \left(\frac{d\sigma}{d\Omega}\right) d\Omega = \frac{\text{Rate of particles scattered into } d\Omega}{\text{Incident flux}}$$

The total cross section $\sigma$ is obviously: $\sigma = \int_{d\Omega} \left(\frac{d\sigma}{d\Omega}\right) d\Omega$

The general SWE is: $$-\frac{\hbar^2}{2\mu}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$ **1.**

Its complex-conjugate is: $$-\frac{\hbar^2}{2\mu}\nabla^2\psi^* + V\psi^* = -i\hbar\frac{\partial\psi^*}{\partial t}$$ **2.**

Exercise for the student: Multiply 1. on the left by $\psi^*$ and 2. on the left by $\psi$ .
Then take the difference of the resulting two equations. The result should be:

$$\frac{\partial}{\partial t}\psi^*\psi = -\frac{\hbar}{2\mu i}(\psi\nabla^2\psi^* - \psi\nabla^2\psi^*)$$

Another exercise: Use some vector calculus to show the above result can be written:

$$\frac{\partial}{\partial t}\psi^*\psi + \nabla\cdot\mathbf{j} = 0$$

Where the “current” **j** is: $$\mathbf{j} = \frac{\hbar}{2\mu i}(\psi^*\nabla\psi - \psi\nabla\psi^*)$$

Are we any closer to reality?? Yes.

Look: The incident wave function was just $A \exp ikz$

Its current density **j** is $\mathbf{j} = \frac{\hbar}{2\mu i}(\psi^* \nabla \psi - \psi \nabla \psi^*)$

$$= \frac{\hbar k}{\mu}|A|^2$$

This is the incident flux.

All that remains now is to get the result for the scattered current density $\mathbf{j}_{sc}$

Recall from page 6: $\psi_{sc} = Af(\theta, \phi)\frac{\exp ikr}{r} \quad \text{as } r \to \infty$

Let's use this to get the scattered current density from $\mathbf{j} = \frac{\hbar}{2\mu i}(\psi^* \nabla \psi - \psi \nabla \psi^*)$.

In spherical coordinates: $$\nabla = \mathbf{u}_r \frac{\partial}{\partial r} + \mathbf{u}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{u}_\phi \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}$$

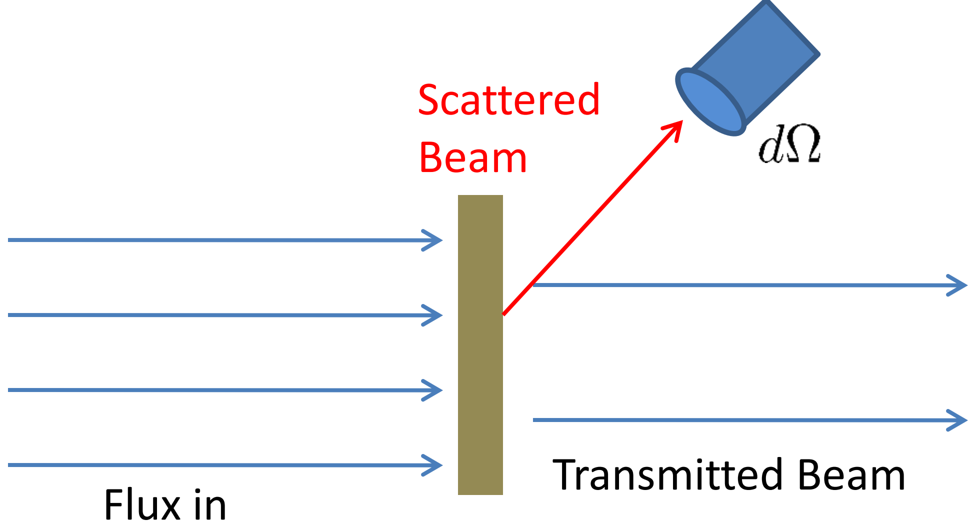
In the limit that $r \longrightarrow \infty$ the only terms that matter to us is the first one.

$$\nabla_{r\to\infty} = \mathbf{u}_r \frac{\partial}{\partial r}$$

Student exercise: Apply this result to the scattered wavefunction $\psi_{sc} = Af(\theta, \phi)\frac{\exp ikr}{r}$ to determine the scattered current as:

$$\mathbf{j}_{sc} = \frac{\hbar k}{\mu r^2}|A|^2|f|^2\mathbf{u}_r$$

This is a **vector** quantity, direct radially outward from the interaction zone. Where does it go experimentally?




It goes into our detector, which occupies a solid angle $d\Omega$ and has an **area** of $r^2\mathbf{u}_r d\Omega$

The **rate** of particles entering the detector is therefore: $\mathbf{j}_{sc} = \frac{\hbar k}{\mu r^2}|A|^2|f|^2\mathbf{u}_r$

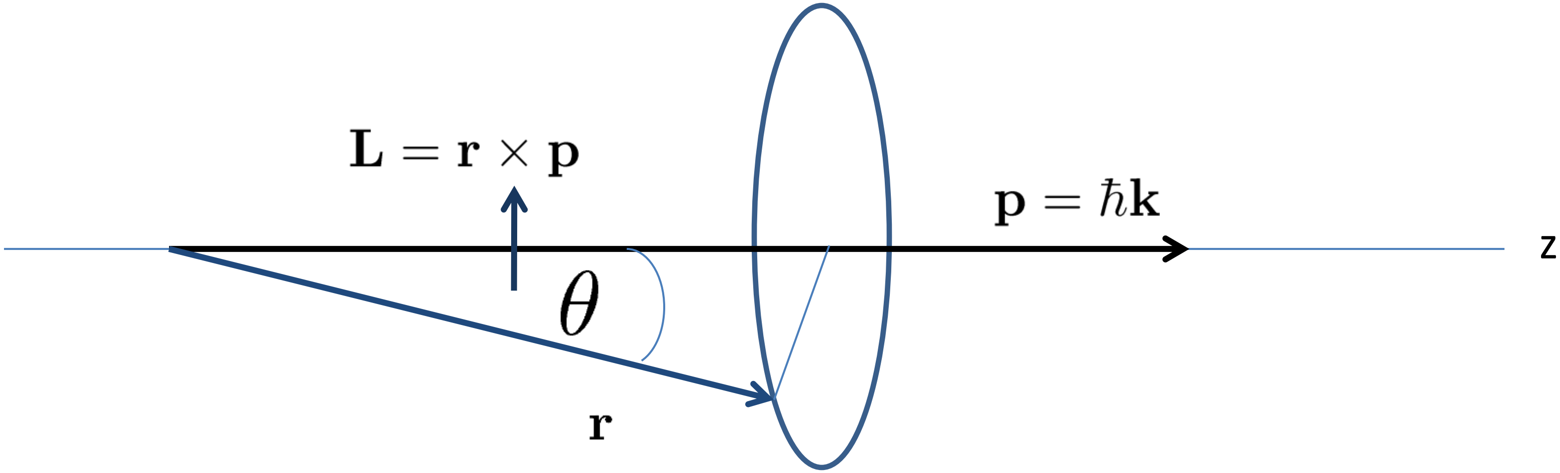
$$\mathbf{j}_{sc} \cdot r^2\mathbf{u}_r d\Omega = \frac{\hbar k}{\mu}|A|^2|f|^2 d\Omega$$

Cross section

$$d\sigma = \left(\frac{d\sigma}{d\Omega}\right) d\Omega = \frac{\text{Rate of particles scattered into } d\Omega}{\text{Incident flux}}$$

$$= \frac{\mathbf{j}_{sc} \cdot \mathbf{u}_r r^2 d\Omega}{j_{inc}} = |f|^2 d\Omega$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = |f(\theta, \phi)|^2$$




The basis functions for the 3-D SWE are the Spherical Bessel functions (radial) and the Spherical Harmonics (angular). These functions form "Complete Sets" and this means any arbitrary function of $(r, \phi, \theta)$ can be expanded in a series representation of these functions.

- Beam comes in along z-axis: plane wave exp ikz
- Angular momentum has ***no*** z-component $\rightarrow m_z = 0$
- Therefore, $Y\ell m(\theta, \phi) \sim P_\ell(cos\theta)$

We therefore write:

$$\exp ikz = \sum_{\ell=0}^{\infty} a_\ell j_\ell(kr) P_\ell(\cos\theta),$$

The task is to determine the expansion coefficients $a_\ell$

Refer to Appendix slides of this lecture for how you can derive that the $a_\ell = (2\ell+1)i^\ell$

We have, therefore: $\psi_{inc} = \exp ikz = \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell j_\ell(kr)P_\ell(\cos\theta),$

$$= \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell \frac{\sin(kr - \ell\pi/2)}{kr}P_\ell(\cos\theta), \quad r \to \infty$$

We need a similar expansion for the total outgoing wave function:

$$\psi \sim \exp ikz + f(\theta,\phi)\frac{\exp ikr}{r} \qquad \textbf{(**)}$$

$$= \sum_{\ell=0}^{\infty} b_\ell \frac{u_\ell(kr)}{kr}P_\ell(\cos\theta)$$

In the asymptotic limit of large r, the function $u_\ell(kr) = \sin(kr - \ell\pi/2 + \delta_\ell)$

$$\sum_{\ell=0}^{\infty} b_\ell \frac{u_\ell(kr)}{kr} P_\ell(\cos\theta) = \sum_{0}^{\infty} b_\ell \frac{\sin(kr - \ell\pi/2 + \delta_\ell)}{kr} P_\ell(\cos\theta)$$

As you will work out for yourself, following the guide in the Appendix, the expansion coefficients are given by: $b_\ell = (2\ell + 1) i^\ell \exp(i\delta_\ell)$

Finally, we have for the outgoing scattered wave:

$$\psi = \frac{1}{2kr} \sum_{\ell=0}^{\infty} (2\ell + 1) i^{\ell+1} \left[ e^{-i(kr - \ell\pi/2)} - e^{2i\delta_\ell} e^{i(kr - \ell\pi/2)} \right] P_\ell(\cos\theta)$$

Remember: the previous mess must be equal to $f(\theta)\frac{\exp ikr}{r}$

$$\sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell}\frac{\sin(kr-\ell\pi/2)}{kr}P_{\ell}(\cos\theta)+f(\theta)\frac{\exp ikr}{r}$$

Taking the difference of the series so that $f(\theta)\frac{\exp ikr}{r}$ is isolated leaves us with:

$$f(\theta)\frac{\exp ikr}{r}=\frac{1}{2kr}\sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell+1}e^{i(kr-\ell\pi/2)}\left[1-e^{2i\delta_{\ell}}\right]P_{\ell}(\cos\theta)$$

$$f(\theta)=\frac{1}{2k}\sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell+1}e^{-i(\ell\pi/2)}\left[1-e^{2i\delta_{\ell}}\right]P_{\ell}(\cos\theta)$$

Note: $i=e^{i\pi/2}\Rightarrow i^{\ell}=e^{i\ell\pi/2}$

$$\sin\delta_{\ell}=\frac{i}{2}\left(e^{-i\delta_{\ell}}-e^{i\delta_{\ell}}\right)\Rightarrow e^{i\delta_{\ell}}\sin\delta_{\ell}=\frac{i}{2}\left(1-e^{2i\delta_{\ell}}\right)$$

$$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta)$$

Differential elastic cross section

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$$

$$= \frac{1}{k^2}\left|\sum_{\ell=0}^{\infty}(2\ell+1)\sin\delta_\ell P_\ell(\cos\theta)\right|^2$$

The $P_\ell$ have the orthogonality condition

$$\int_{d\Omega} P_\ell(\cos\theta)P^*_{\ell'}(\cos\theta) = \frac{4\pi}{2n+1}\delta_{\ell\ell'}$$

Total Elastic cross section:

$$\sigma = \int\left(\frac{d\sigma}{d\Omega}\right)d\Omega = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell$$

We can also define an expression for a reaction cross section.

We had, as before, that the total outgoing wave function is:

$$\psi = \frac{1}{2kr}\sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell+1}\left[e^{-i(kr-\ell\pi/2)} - e^{2i\delta_\ell}e^{i(kr-\ell\pi/2)}\right]P_\ell(\cos\theta)$$

Current density: $$\mathbf{j} = \frac{\hbar}{2\mu i}\left(\psi^*\frac{\partial\psi}{\partial r} - \psi\frac{\partial\psi^*}{\partial r}\right)$$

$$= \frac{\hbar}{4\mu k r^2}\left\{\left|\sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell+1}e^{i\ell\pi/2}P_\ell(\cos\theta)\right|^2 - \left|\sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell+1}e^{-i\ell\pi/2}e^{2i\delta_\ell}P_\ell(\cos\theta)\right|^2\right\}$$

Recall from last lecture: $$j_{inc} = \frac{\hbar k}{m}$$

And also (page 25, L 5) $$d\sigma = \frac{\mathbf{j}_{sc}\cdot\mathbf{u}_r r^2 d\Omega}{j_{inc}}$$

And also $$\int_{d\Omega} P_\ell(\cos\theta)P_{\ell'}(\cos\theta) = \tfrac{4\pi}{2n+1}\delta_{\ell\ell'}$$

Integrating $d\sigma = \frac{\mathbf{j}_{sc} \cdot \mathbf{u}_r r^2 d\Omega}{j_{inc}}$ over the sphere will finally result in:

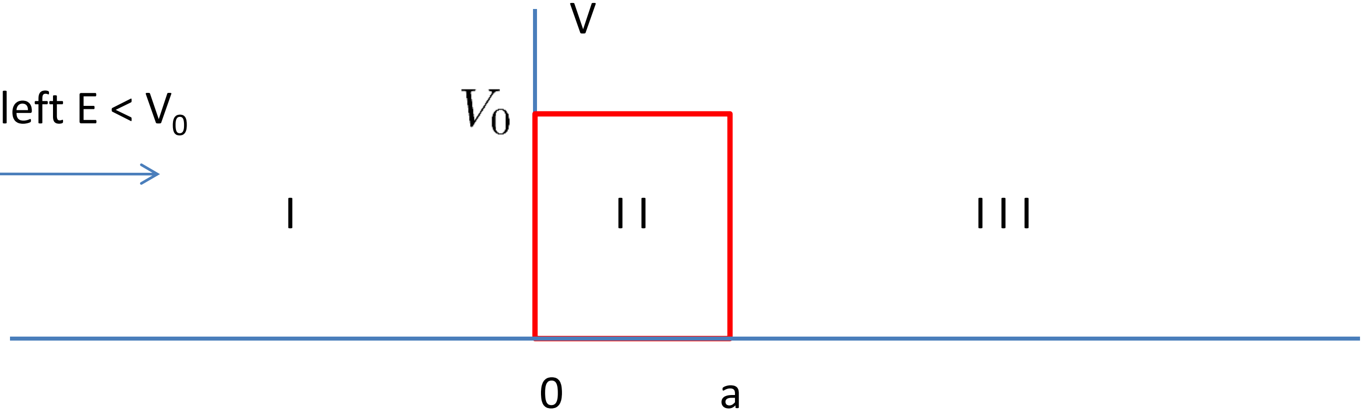
Reaction Cross Section: $$\sigma_{re} = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell + 1) \left(1 - \left|e^{2i\delta_\ell}\right|^2\right)$$

Note: if the phase shift of the scattered wave is real, then no reactions occur. Therefore, for nuclear reactions we require that $\delta_\ell$ be a complex number.

For our purposes in reaction rates, these expressions are not very practical. But what *is* important to note about them is the $k^{-2} \propto E^{-1}$ dependence.

$$\boxed{\sigma_{re} \propto \frac{1}{E}}$$

Wave from left $E < V_0$




$$\psi_I = Ae^{ikx} + Be^{-ikx} \qquad k = \sqrt{2mE}/\hbar$$

$$\psi_{II} = Fe^{qx} + Ge^{-qx} \qquad q = \sqrt{2m(V_0 - E)}/\hbar$$

$$\psi_{III} = Ce^{ikx} \qquad k = \sqrt{2mE}/\hbar$$

At the boundaries x = 0 and x = a, the value of the wave function and its derivatives there must match

$$A + B = F + G$$ At boundary x = 0

$$ik(A - B) = q(F - G)$$ Derivative at boundary x = 0

$$Fe^{qa} + Ge^{-qa} = Ce^{ika}$$ At boundary x = a

$$q(Fe^{qa} + Ge^{-qa}) = ikCe^{ika}$$ Derivative at boundary x = a

Transmission coefficient T is given by, $T = \frac{j_{x>a}}{j_{inc}}$

Exercise for you guys: use these equations to determine transmission coefficient as

$$T = \frac{|C|^2}{|A|^2} = \frac{1}{1 + \frac{k^2+q^2}{2kq}\ \sinh^2 qa}$$

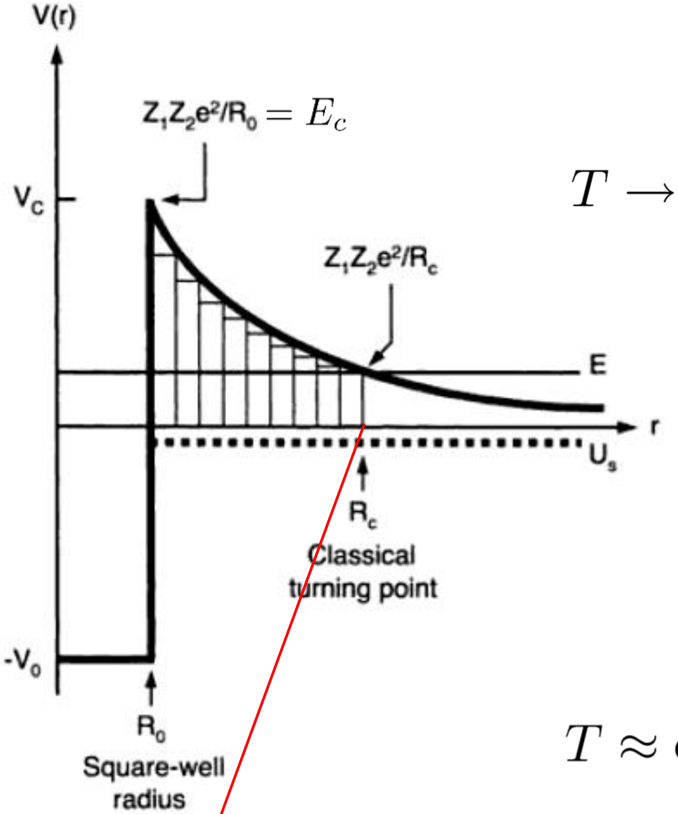
More algebra on this:

$$T = \frac{|C|^2}{|A|^2} = \frac{1}{1 + \frac{k^2+q^2}{2kq}\ \sinh^2 qa}$$

First:

$$\sinh qa = \frac{e^{qa} - e^{-qa}}{2} \to \frac{e^{qa}}{2}, \quad qa \gg 1$$

$$T \to \left(\frac{4kq}{k^2 + q^2}\right)^2 e^{-2qa} \propto e^{-2qa}$$




$$T \to \prod_i^n T_i \approx \exp\left[-\frac{2}{\hbar}\sum_i \sqrt{2\mu(V_i - E)}(R_{i+1} - R_i)\right]$$

$$\underset{n\to\infty}{\longrightarrow} \exp\left[-\frac{2}{\hbar}\int_{R_0}^{R_c} \sqrt{2\mu[V(r) - E]}dr\right]$$

$$T \approx \exp\left[-\frac{2}{\hbar}\sqrt{2\mu}\int_{R_0}^{R_c}\sqrt{\frac{Z_1Z_2e^2}{r} - E}\ dr\right]$$

$$E = Z_1Z_2e^2/R_c$$

After a lot of work: $$\frac{4Z_1Z_2e^2}{\hbar v}\left[\frac{\pi}{2} - \left(\frac{E}{E_c}\right)^{1/2}\left(1 - \frac{E}{E_c}\right)^{1/2} - \arcsin\left(\frac{E}{E_c}\right)\right]$$

Where: $$E_c = Z_1Z_2e^2/R_0$$

Small at stellar energies

Dominant term is just the constant factor of $\pi/2$

In leading order, transmission coefficient is: $T \propto \exp\left(-\frac{2\pi Z_1 Z_2 e^2}{\hbar v}\right)$

And don't forget that $v$ is relative velocity between particles 1 and 2.

Looking back at page 10, we remember the cross section is $\propto E^{-1}$

$$\sigma(E) = \frac{S(E)}{E}\exp\left(-\frac{2\pi Z_1 Z_2 e^2}{\hbar v}\right) = \frac{S(E)}{E}\exp\left[-2\pi\eta(v)\right]$$

S(E) is called the "Astrophysical S-Factor". It is a function that "absorbs" all of the fine details that our approximations have omitted.

With this parameterization of the cross section, we factor out the 1/E dependence and the very strong s-wave penetrability factor, which tend to dominate the cross section at low incident energy.

Now, let's use the previous result for the cross section in our rate formula!

Remember from L6, page 28:

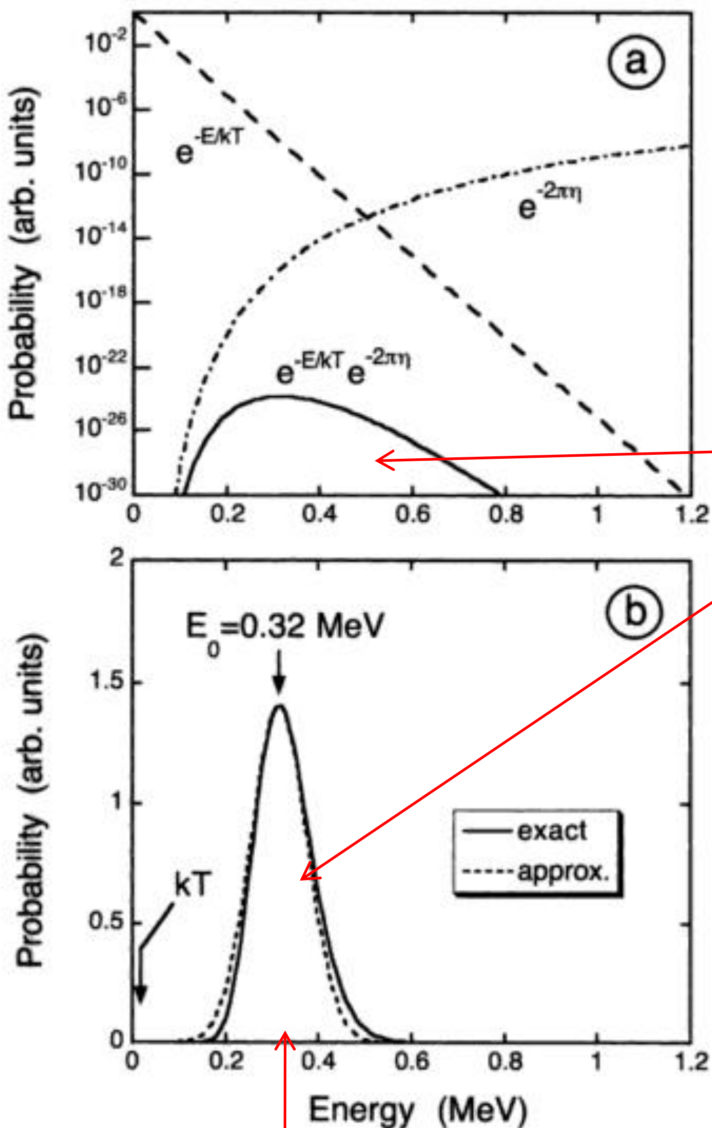
$$r_{12} = \left(\frac{8}{\pi\mu}\right)^{1/2} \frac{N_1 N_2}{1+\delta_{12}} \tau^{-3/2} \int_0^\infty E_{12}\sigma_{12}(v) \exp\left(-\frac{E_{12}}{\tau}\right) dE_{12}$$

Subbing in the cross section formula

$$r_{12} = \left(\frac{8}{\pi\mu}\right)^{1/2} \frac{N_1 N_2}{1+\delta_{12}} \tau^{-3/2} \int_0^\infty S(E_{12}) \exp\left(-\frac{b}{\sqrt{E_{12}}} - \frac{E_{12}}{\tau}\right) dE_{12}$$

Where: $b = 2\pi \frac{Z_1 Z_2 e^2}{\hbar} \left(\frac{\mu}{2}\right)^{1/2} = 31.27 Z_1 Z_2 \mu^{1/2} \ \ \mathrm{keV}^{1/2}$

We also define the Sommerfeld Parameter as: $\eta(v) = \frac{Z_1 Z_2 e^2}{\hbar v}$




Log scale plot

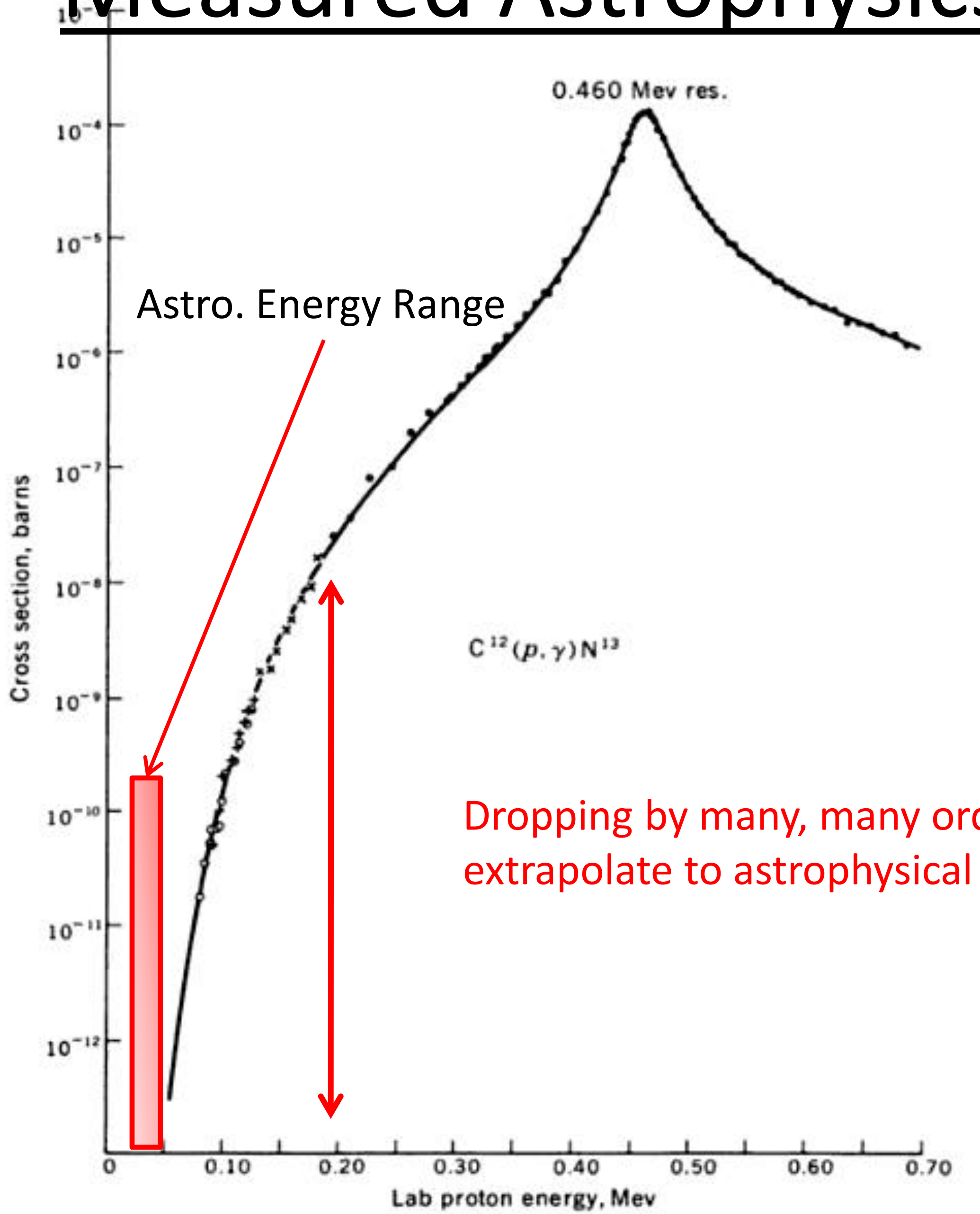
This is where the action happens in thermonuclear burning!

$$\exp\left(-\frac{b}{\sqrt{E_{12}}}-\frac{E_{12}}{\tau}\right)$$

Linear scale plot

This curve (integrand) is called the ***Gamow Window***

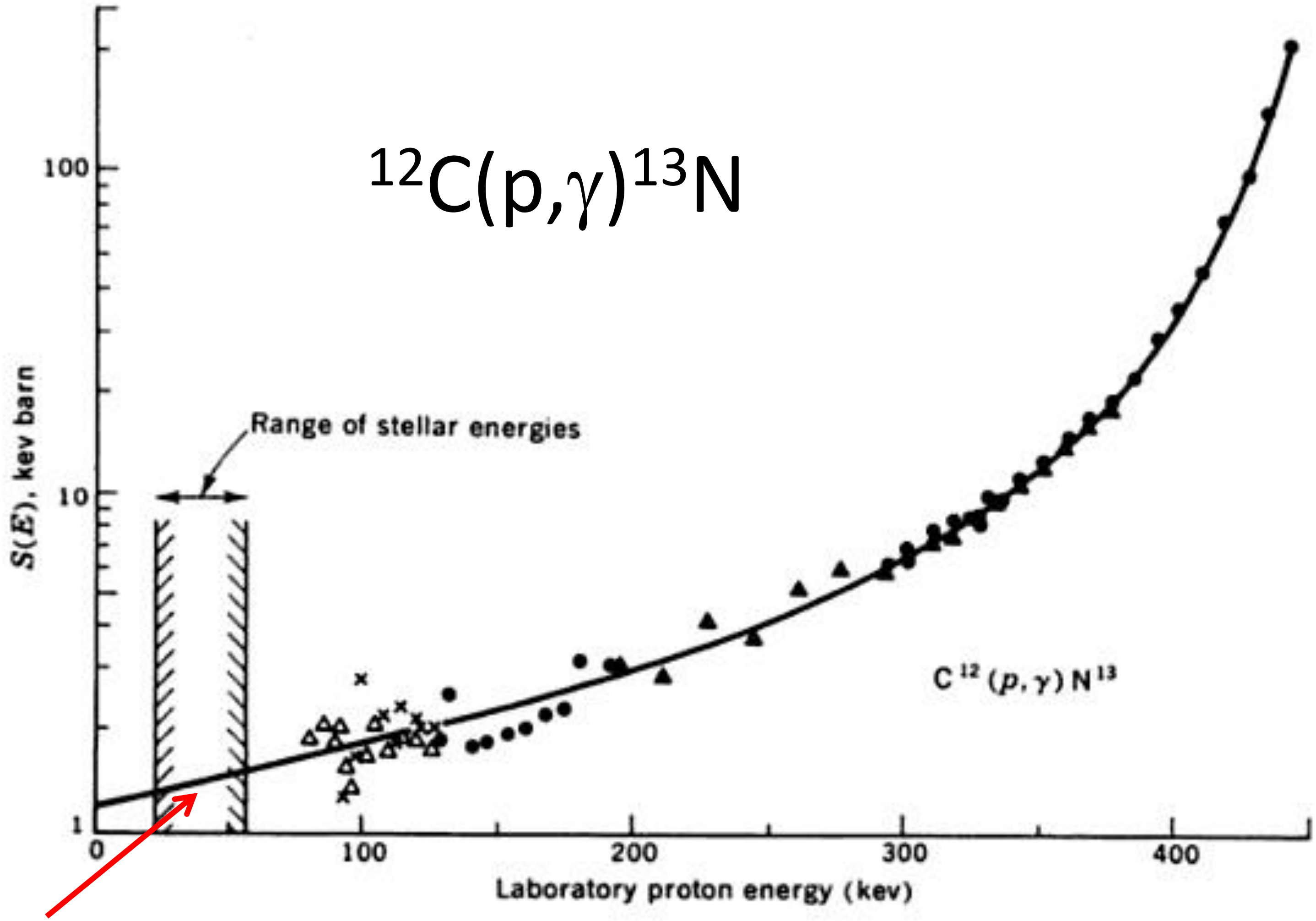
# Measured Astrophysics Cross-Section




$^{12}C(p,\gamma)^{13}N$

Dropping by many, many orders! How to extrapolate to astrophysical energies?!

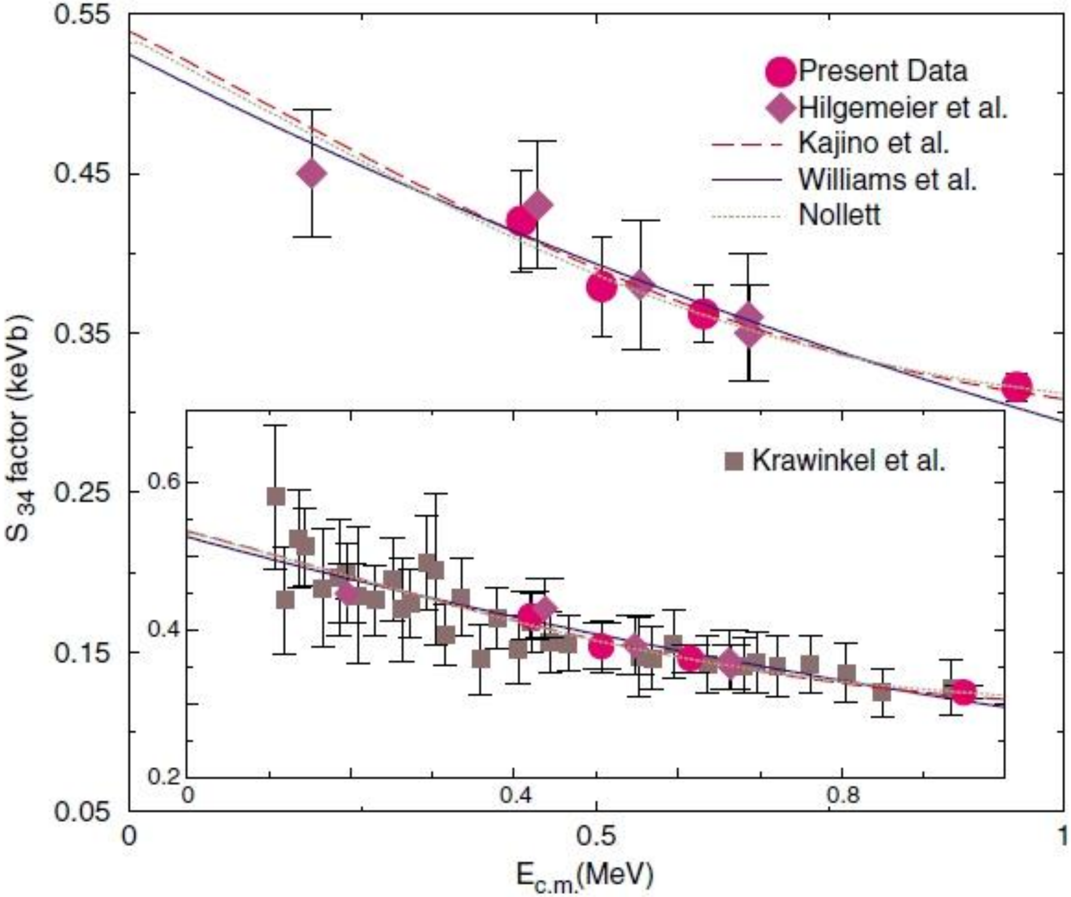
# S-Factor of Previous Data




Can be parameterized by a simple linear function of Energy

# A Reaction in the Solar PP Chain

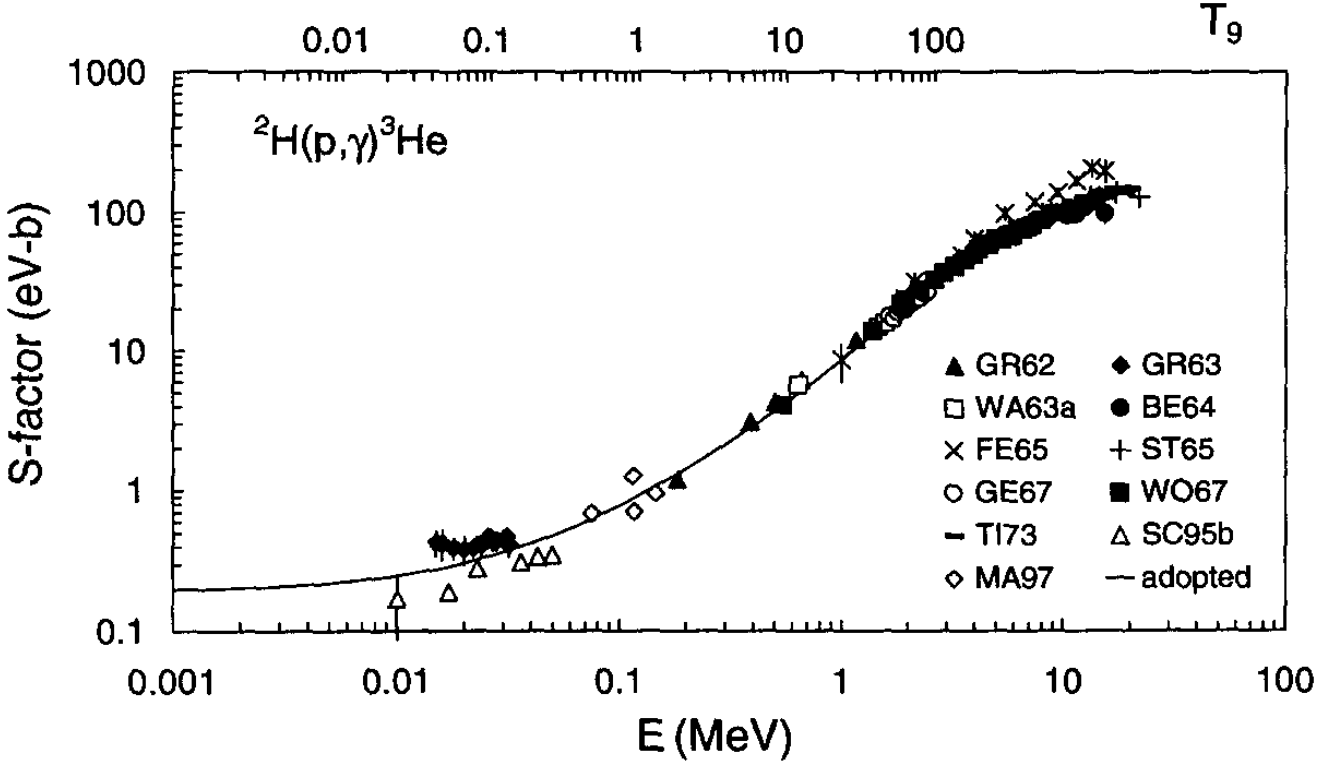
## $^3He(\alpha,\gamma)^7Be$




B. S. Nara Singh et al., PRL **93, 262503 (2004)**

# Another Reaction in the Solar PP Chain

## $^{2}H(p,\gamma)^{3}He$



We now see the utility of using the S-factor formalism.

It allows us to find a constant, or linear function, by which to parameterize the experimentally determined cross-section *within the range of astrophysical energy* for the reaction under consideration.

The Objective of (some) nuclear astrophysics experiments, then, is this:

$$\sigma(E) = S(E)\frac{1}{E}\exp\left(-\frac{2\pi Z_1 Z_2 e^2}{\hbar v}\right)$$

Measure this with a reaction experiment as far down in energy as possible (as close as possible to stellar energy)

This is well defined, by the experimental conditions (E is known, and therefore, so is v).

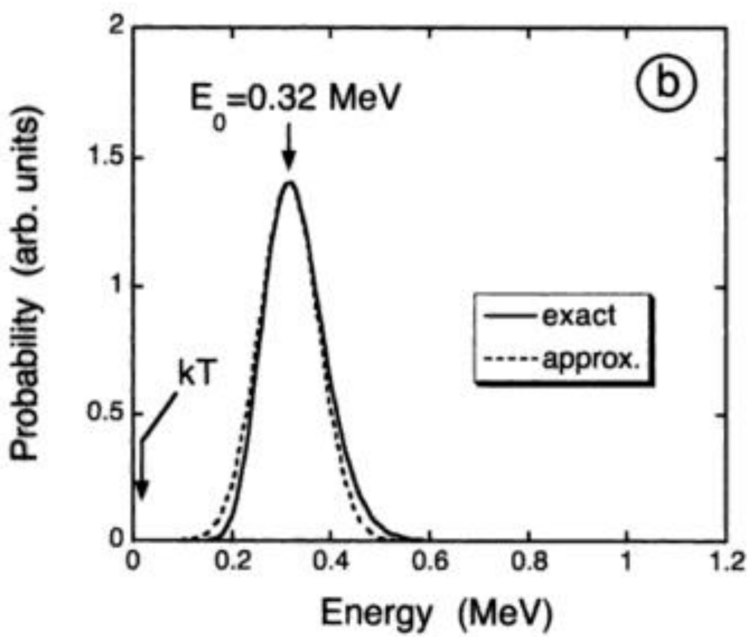
S(E) is a simple function at, or near, astro. Energies, and can then be "safely" extrapolated down to the stellar energy range.

Let's return to the reaction rate formula:

$$r_{12} = \left(\frac{8}{\pi\mu}\right)^{1/2} \frac{N_1 N_2}{1+\delta_{12}} \tau^{-3/2} \int_0^\infty S(E_{12}) \exp\left(-\frac{b}{\sqrt{E_{12}}} - \frac{E_{12}}{\tau}\right) dE_{12}$$

Let's focus on the integral, treating S(E) as a constant in lowest order. Drop the "12" subscripts for clarity in what follows.

$$\int_0^\infty \exp\left(-\frac{b}{\sqrt{E}} - \frac{E}{\tau}\right) dE$$




The exponential function in the integrand is shown here by the solid line.

It looks very similar to something like a Gaussian.

Let's use this fact to motivate an approximation of this integral.

How can we cast the integrand of the previous page into a form that is something like a Gaussian?

Take the integrand $F(E) = \exp\left(-\frac{b}{\sqrt{E}} - \frac{E}{\tau}\right)$ $\tau \equiv kT$

Take the derivative $dF/dE = 0$ to determine the value of the energy, E, at the location of the maximum. Call this value of energy, $E_{\text{eff}}$ , the "effective" burning energy.

You should get: $$E_{\text{eff}} = \left(\frac{b\tau}{2}\right)^{2/3} = 1.22(Z_1^2 Z_2^2 \mu T_6^2)^{1/3} \ \text{keV}$$

Note: $T_6$ means the temperature is in units of $10^6$ kelvin

Evaluate F(E) at this value of $E_{\text{eff}}$ to determine the value of the function at maximum.

$$F_{\text{max}} = \exp\left(-\frac{3E_{\text{eff}}}{\tau}\right)$$

Now, we're ready to construct the Gaussian. Set:

$$\exp\left(-\frac{b}{\sqrt{E}}-\frac{E}{\tau}\right) \approx F_{\rm max}\exp\left[-\left(\frac{E-E_{\rm eff}}{\Delta/2}\right)^2\right]$$

The width of our Gaussian is $\Delta$ and it must be determined to complete the approximation.

We determine $\Delta$ by demanding that the curvature of the two functions above match at $E = E_{\rm eff}$ .

This is equivalent to requiring the 2nd derivatives at $E = E_{\rm eff}$ be the same. This is for you to do. You should end up with:

$$\Delta = \frac{4}{\sqrt{3}}(E_{\rm eff}\;\tau)^{1/2} = 0.75(Z_1^2 Z_2^2 \mu T_6^5)^{1/6}$$

Thermonuclear burning happens within the energy range $E_{\rm eff} \pm \Delta/2$

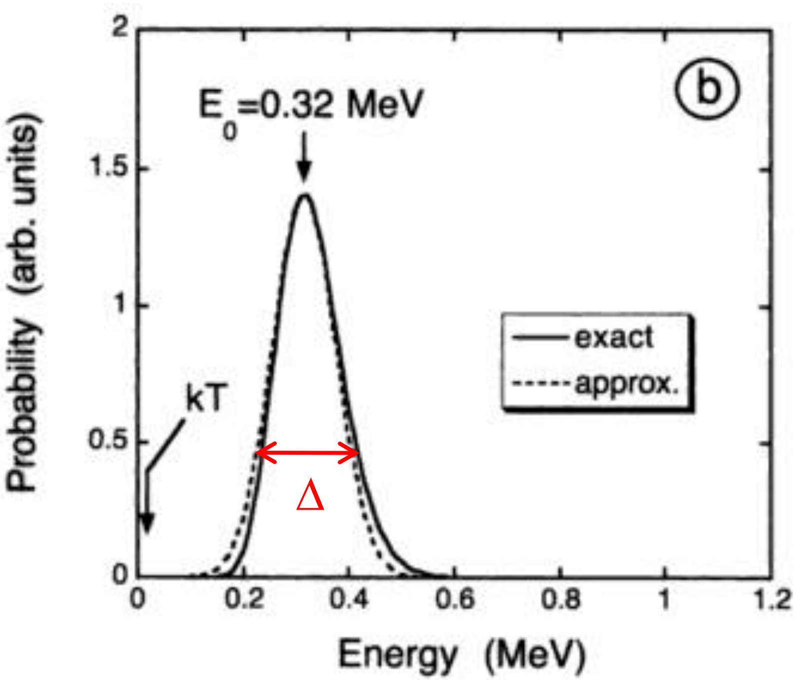
# Summary Thus Far:

Charged particle thermonuclear burning happens around the relative kinetic energy value of:

$$E_{\text{eff}} = \left(\frac{b\tau}{2}\right)^{2/3} = 1.22(Z_1^2 Z_2^2 \mu T_6^2)^{1/3} \;\; \text{keV}$$

The range of kinetic energy over which the burning occurs is the width of our Gaussian approximation for the integrand of the rate. The width of this Gaussian is given by:

$$\Delta = \frac{4}{\sqrt{3}}(E_{\text{eff}}\;\tau)^{1/2} = 0.75(Z_1^2 Z_2^2 \mu T_6^5)^{1/6}$$




When using these formulae, use integers for the charges Z and be sure the temperature is scaled in units of $10^6$. Also, use atomic mass values in $\mu$ ; for example: 4.004 for the mass of $^4$He.

Returning to the rate formula, we insert our Gaussian approximation in the integral and we pull out the S-Factor, treating it as a constant.

$$r_{12} = \left(\frac{8}{\pi\mu}\right)^{1/2} \frac{N_1 N_2}{1+\delta_{12}} \tau^{-3/2} S_0 F_{\max} \int_0^\infty \exp\left[-\left(\frac{E - E_{\text{eff}}}{\Delta/2}\right)^2\right] dE$$

The integral, being a Gaussian, can be extended to $-\infty$ with tiny, tiny error.

The value of the integral then becomes: $\sqrt{\pi}\Delta/2$

$$r_{12} = \left(\frac{2}{\mu}\right)^{1/2} \frac{N_1 N_2}{1+\delta_{12}} \tau^{-3/2} \Delta S_0 F_{\max}$$

$$r_{12} = \left(\frac{2}{\mu}\right)^{1/2} \frac{N_1 N_2}{1+\delta_{12}} \tau^{-3/2} \Delta S_0 \exp(-3E_{\text{eff}}/\tau)$$

Next lecture we will use this last result to express the reaction rate as a power law in temperature.

This will give us a qualitative understanding of the sensitivity of the reaction rate with stellar temperature.

# Assignment: Part I

Start with the *Ansatz*: $$\frac{dP}{dr} = -\frac{4\pi}{3}G\rho_c^2 \; r \;\; \exp\left[-\left(\frac{r}{a}\right)^2\right]$$

a) Determine where the pressure gradient changes sign. $\frac{d^2P}{dr^2} = 0$

b) Determine the Pressure, P(r), as a function of radial coordinate.

c) Determine the Stellar Mass as a function of radial coordinate: M(r). Hint: You need to use both the equation for conservation of mass and hydrostatic equil'm equation:

$$\frac{dM_r}{dr} = 4\pi r^2 \rho_r \qquad\qquad \frac{dP_r}{dr} = -G\frac{M_r}{r^2}\rho_r$$

d) Determine the density function $\rho(r)$. Confirm if it has the correct limiting behaviour as $r \rightarrow 0$.

<u>Due by ??, 2009</u>

Some steps and hints for deriving the partial wave formulas used in this lecture.

# APPENDIX TO LECTURE 6

How to show the result for the plane wave expansion?

$$\exp ikz = \sum_{\ell=0}^{\infty} (2\ell+1) i^{\ell} j_{\ell}(kr) P_{\ell}(\cos\theta), \qquad z = r\cos\theta$$

Star t with the expansion: $$\exp ikz = \sum_{\ell=0}^{\infty} a_{\ell} j_{\ell}(kr) P_{\ell}(\cos\theta) \qquad \textbf{(1)}$$

We can do this because the spherical Bessel Functions and the Legendre polynomials form a complete set → any arbitrary function can be expanded in the basis of these functions.

The task is to show that: $a_{\ell} = (2\ell+1) i^{\ell}$

You will need the orthogonality condition:

$$\int_0^{\pi} P_{\ell}(\cos\theta) P_m(\cos\theta) \sin\theta d\theta = \frac{2}{2n+1} \delta_{\ell m}$$

You will also need the power series representation for the Bessel Function:

$$j_\ell(x) = \sqrt{\pi}\sum_{s=0}^{\infty}\frac{(-1)^s x^{2s+\ell}}{s!(s+\ell+1/2)!2^{2s+\ell+1}}$$

And you will also need to know, as can be shown from the Gamma Function:

$$\left(n+\frac{1}{2}\right)! = \frac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi}$$

The steps are as follows:

1. Use the orthogonality condition on equation (1) to solve for $a_\ell j_\ell(kr)$
2. Differentiate the formula obtained in step 1 $\ell$-times with respect to (kr) and then set r = 0 to eliminate the r-dependence. Use the power series above to evaluate the $\ell^{th}$ derivative of the Bessel Function.
3. Evaluate the remaining integral using:

$$\int_0^\pi \cos^\ell\theta P_\ell(\cos\theta)\sin\theta d\theta = \frac{2^{2\ell+1}\ell!\,\ell!}{2\ell+1}$$

Partial wave expansion for the total wave function:

1. Start with the partial wave expansion for the incident free particle wave function on page 5, $\psi_{inc}$

2. Substitute the expansion into equation (**) on page 5 and isolate $f(\theta)\frac{\exp ikr}{r}$ by grouping the two infinite series together on one side of the equation.

3. Use $\sin x = \frac{i}{2}\left[\exp(-ix) - \exp(ix)\right]$ for the sine factors in the series.

4. In the series, group the exponentials according to exp(ikr) and exp(-ikr)

5. Argue to yourself that the terms having exp(-ikr) must die in the series. (hint: the function in step 2 above only has outgoing waves)

6. From step 5, the expansion coefficients will be determined.