Nuclear Astrophysics

Lecture 6

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Thermonuclear Reaction Rate in Stars

THE ROAD TO NUCLEAR REACTION RATES

Some basic kinematics: We have two particles with masses m_1 and m_2 with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$

The velocity of their common centre of mass is: $\mathbf{V} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1+m_2}$

The velocity of particle 1 relative to the CoM velocity is just:

$$\mathbf{v}_1 - \mathbf{V} = \frac{m_2}{m_1+m_2}(\mathbf{v}_1 - \mathbf{v}_2) = \frac{m_2}{m_1+m_2}\mathbf{v}$$

And **v** is just the relative velocity between 1 and 2.

Similarly, particle 2 has a velocity relative to CoM velocity:

$$\mathbf{v}_2 - \mathbf{V} = -\frac{m_1}{m_1+m_2}\mathbf{v}$$

Before the collision, the total incident kinetic energy is:

$$T_i = \frac{1}{2}(m_1 v_1^2 + m_2 v_2^2)$$

Using the previous two vector equations, we can substitute in for v_1 and v_2 in terms of v and V. (An exercise for you)

$$T_i = \frac{1}{2}(MV^2 + \mu v^2)$$

$$M = m_1 + m_2$$

$$\mu = m_1 m_2/(m_1 + m_2)$$

The first term is the kinetic energy of the center of mass itself; while the second term is the kinetic energy of the reduced mass **as it moves in the center of mass frame**.

Nuclear reaction rate: The reaction rate is proportional to the number density of particle species 1, the flux of particle species 2 that collide with 1, and the reaction cross section.

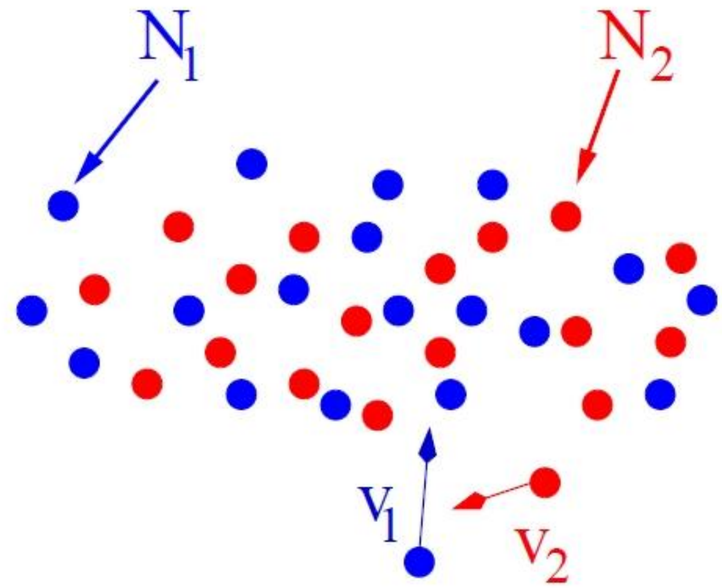
Flux of N_2 as seen by N_1 : $N_2 v$

Flux of N_1 as seen by N_2 : $N_1 v$

Reaction cross section: $\sigma(v)$

$$r_{12} = N_1 N_2 v \sigma(v)$$

This v is the **relative** velocity between the two colliding particles.



Important: this reaction rate formula only holds when the flux of particles has a mono-energetic velocity distribution of just v

Inside a star, the particles clearly do not move with a mono-energetic velocity distribution. Instead, they have their own velocity distributions.

We must generalize the previous rate formula for the stellar environment. From Lecture 2,3 the particles 1 and 2 will have velocity *distributions* given by Maxwell-Boltzmann distributions. We have the 6-D integral:

$$r_{12} = \int_{d^3 v_1} \int_{d^3 v_2} N_1(v_1) N_2(v_2) v \sigma(v) d^3 v_1 d^3 v_2$$

The *fraction* of particles 1 with velocities between v_1 and $v_1 + dv_1$ is therefore,

$$N_1(v_1) dv_{1_x} dv_{1_y} dv_{1_z} = N_1 \left(\frac{m_1}{2\pi\tau} \right)^{3/2} \exp(-m_1 v_1^2 / 2\tau) \, dv_{1_x} dv_{1_y} dv_{1_z}$$

And similarly for particle species 2.

Let's take a closer look at:

$$\exp(-m_1 v_1^2 / 2\tau) \exp(-m_2 v_2^2 / 2\tau)$$

$$\exp(-m_1 v_1^2/2\tau)\exp(-m_2 v_2^2/2\tau) = \exp(-[m_1 v_1^2 + m_2 v_2^2]/2\tau)$$

From equations on page 4, we can write the argument in [...] in terms of the center of mass velocity V and relative velocity v .

$$\exp(-[m_1 v_1^2 + m_2 v_2^2]/2\tau) = \exp\left(-\frac{(m_1+m_2)V^2}{2\tau} - \frac{\mu v^2}{2\tau}\right)$$

So in terms of the CoM parameters,

$$N_1(v_1)N_2(v_2)d^3v_1 d^3v_2 = N_1 N_2 \frac{(m_1 m_2)^{3/2}}{(2\pi\tau)^3}\exp\left(-\frac{(m_1+m_2)V^2}{2\tau} - \frac{\mu v^2}{2\tau}\right)d^3v_1 d^3v_2$$

The reaction rate now becomes (6-D integral):

$$r_{12} = N_1 N_2 \frac{(m_1 m_2)^{3/2}}{(2\pi\tau)^3}\int_{d^3v_1}\int_{d^3v_2} v\sigma(v)\exp\left(-\frac{(m_1+m_2)V^2}{2\tau} - \frac{\mu v^2}{2\tau}\right)d^3v_1 d^3v_2$$

And we note: $V^2 = V_x^2 + V_y^2 + V_z^2$ and $v^2 = v_x^2 + v_y^2 + v_z^2$

We now need to change the differential variables into the new CoM variables.

$$d^3v_1 d^3v_2 \equiv dv_{1_x} dv_{1_y} dv_{1_z} dv_{2_x} dv_{2_y} dv_{2_z}$$

$$= (dv_{1_x} dv_{2_x})(dv_{1_y} dv_{2_y})(dv_{1_z} dv_{2_z})$$

From page 3, in component form, we have:

$$v_{1_x} = \frac{m_2}{m_1+m_2} v_x + V_x \qquad v_{2_x} = \frac{-m_1}{m_1+m_2} v_x + V_x$$

Jacobian:

$$\left| \begin{array}{cc} \frac{\partial v_{1_x}}{\partial V_x} & \frac{\partial v_{1_x}}{\partial v_x} \\ \frac{\partial v_{2_x}}{\partial V_x} & \frac{\partial v_{2_x}}{\partial v_x} \end{array} \right| = \left| \begin{array}{cc} 1 & \frac{m_2}{m_1+m_2} \\ 1 & \frac{-m_1}{m_1+m_2} \end{array} \right| = -1$$

And this is the same for the case of y and z components.

The rate integral now becomes:

$$r_{12} = N_1 N_2 \frac{(m_1 m_2)^{3/2}}{(2\pi\tau)^3} \int_{d^3v_1} \int_{d^3v_2} v\sigma(v) \exp\left(-\frac{(m_1+m_2)V^2}{2\tau} - \frac{\mu v^2}{2\tau}\right) d^3v_1 d^3v_2$$

$$= N_1 N_2 \frac{(m_1 m_2)^{3/2}}{(2\pi\tau)^3} \int_{d^3V} \exp\left(-\frac{(m_1+m_2)V^2}{2\tau}\right) d^3V \int_{d^3v} v\sigma(v) \exp\left(-\frac{\mu v^2}{2\tau}\right) d^3v$$

$$= \quad N_1 N_2 \left(\frac{m_1+m_2}{2\pi\tau}\right)^{3/2} \int_{d^3V} \exp\left(-\frac{(m_1+m_2)V^2}{2\tau}\right) d^3V \times$$
$$\left(\frac{\mu}{2\pi\tau}\right)^{3/2} \int_{d^3v} v\sigma(v) \exp\left(-\frac{\mu v^2}{2\tau}\right) d^3v$$

$$= N_1 N_2 \left(\frac{\mu}{2\pi\tau}\right)^{3/2} 4\pi \int_0^\infty v^3 \sigma(v) \exp\left(-\frac{\mu v^2}{2\tau}\right) dv$$

$$r_{12} = 4\pi N_1 N_2 \left(\frac{\mu}{2\pi\tau}\right)^{3/2} \int_0^\infty v^3 \sigma(v) \exp\left(-\frac{\mu v^2}{2\tau}\right) dv$$

Note: the product N_1N_2 is the number of unique particle pairs (per unit volume). If it should happen that 1 and 2 are the same species, then we must make a small correction to the rate formula to avoid double-counting of particle pairs.

$$r_{12} = 4\pi \frac{N_1 N_2}{1+\delta_{12}} \left(\frac{\mu}{2\pi\tau}\right)^{3/2} \int_0^\infty v^3 \sigma(v) \exp\left(-\frac{\mu v^2}{2\tau}\right) dv$$

$$= \frac{N_1 N_2}{1+\delta_{12}} \langle \sigma v \rangle$$

$\delta_{12} =$ Kronecker delta

$$\langle \sigma v \rangle \equiv 4\pi \left(\frac{\mu}{2\pi\tau}\right)^{3/2} \int_0^\infty v^3 \sigma(v) \exp\left(-\frac{\mu v^2}{2\tau}\right) dv$$

We can extend the previous result to the case when one of the particles in the entrance channel is a photon. So reaction is: $1 + \gamma \rightarrow 2 + 3$

The rate: $r_{1\gamma} = N_1 N_\gamma v \sigma$

As before, we generalize this by integrating over the number density distributions: A Maxwell-Boltzmann for species 1, and for photons we recall from Lecture 2,3 the following:

$$\frac{U_\omega}{V} d\omega = u_\omega d\omega = \frac{\pi\hbar}{(\pi c)^3} \frac{\omega^3}{\exp(\hbar\omega/\tau) - 1} d\omega$$

$$= \frac{\pi}{(\hbar\pi c)^3} \frac{E_\gamma^3}{\exp(E_\gamma/\tau) - 1} dE_\gamma$$

Number of photons per unit volume between E_γ and $E_\gamma + dE_\gamma$:

$$N(E_\gamma) dE_\gamma = \frac{\pi}{(\hbar\pi c)^3} \frac{E_\gamma^2}{\exp(E_\gamma/\tau) - 1} dE_\gamma$$

The Einstein postulate of Special Relativity: speed of light is the same in **all** reference frames. Therefore, the relative velocity $v = c$

$$r_{1\gamma} = N_1 N_\gamma v \sigma \rightarrow \quad r_{1\gamma} = \frac{\pi c}{(\pi\hbar c)^3} \int_{d^3 v_1} \int_0^\infty N_1(v_1)\sigma(E_\gamma) \frac{E_\gamma^2}{\exp(E_\gamma/\tau) - 1} dE_\gamma d^3 v_1$$

$$= \frac{\pi c}{(\pi\hbar c)^3} \int_{d^3 v_1} N_1(v_1) d^3 v_1 \int_0^\infty \sigma(E_\gamma) \frac{E_\gamma^2}{\exp(E_\gamma/\tau) - 1} dE_\gamma$$

= N_1 because N_1 is M-B

$$= \frac{8\pi N_1}{h^3 c^2} \int_0^\infty \sigma(E_\gamma) \frac{E_\gamma^2}{\exp(E_\gamma/\tau) - 1} dE_\gamma$$

And $\sigma(E_\gamma)$ is the photo-disintegration cross section.

Reaction Rate Summary

Reaction rate for charged particles: $1 + 2 \rightarrow 3 + 4$

$$r_{12} = \frac{4\pi N_1 N_2}{1+\delta_{12}} \left(\frac{\mu}{2\pi\tau}\right)^{3/2} \int_0^\infty v^3 \sigma(v) \exp\left(-\frac{\mu v^2}{2\tau}\right) dv$$

$$r_{12} = \left(\frac{8}{\pi\mu}\right)^{1/2} \frac{N_1 N_2}{1+\delta_{12}} \tau^{-3/2} \int_0^\infty E_{12} \sigma_{12}(v) \exp\left(-\frac{E_{12}}{\tau}\right) dE_{12}$$

Reaction rate for photodisintegration (photon in entrance channel): $1 + \gamma \rightarrow 2 + 3$

$$r_{1\gamma} = \frac{8\pi N_1}{h^3 c^2} \int_0^\infty \sigma_{1\gamma}(E_\gamma) \frac{E_\gamma^2}{\exp(E_\gamma/\tau) - 1} dE_\gamma$$