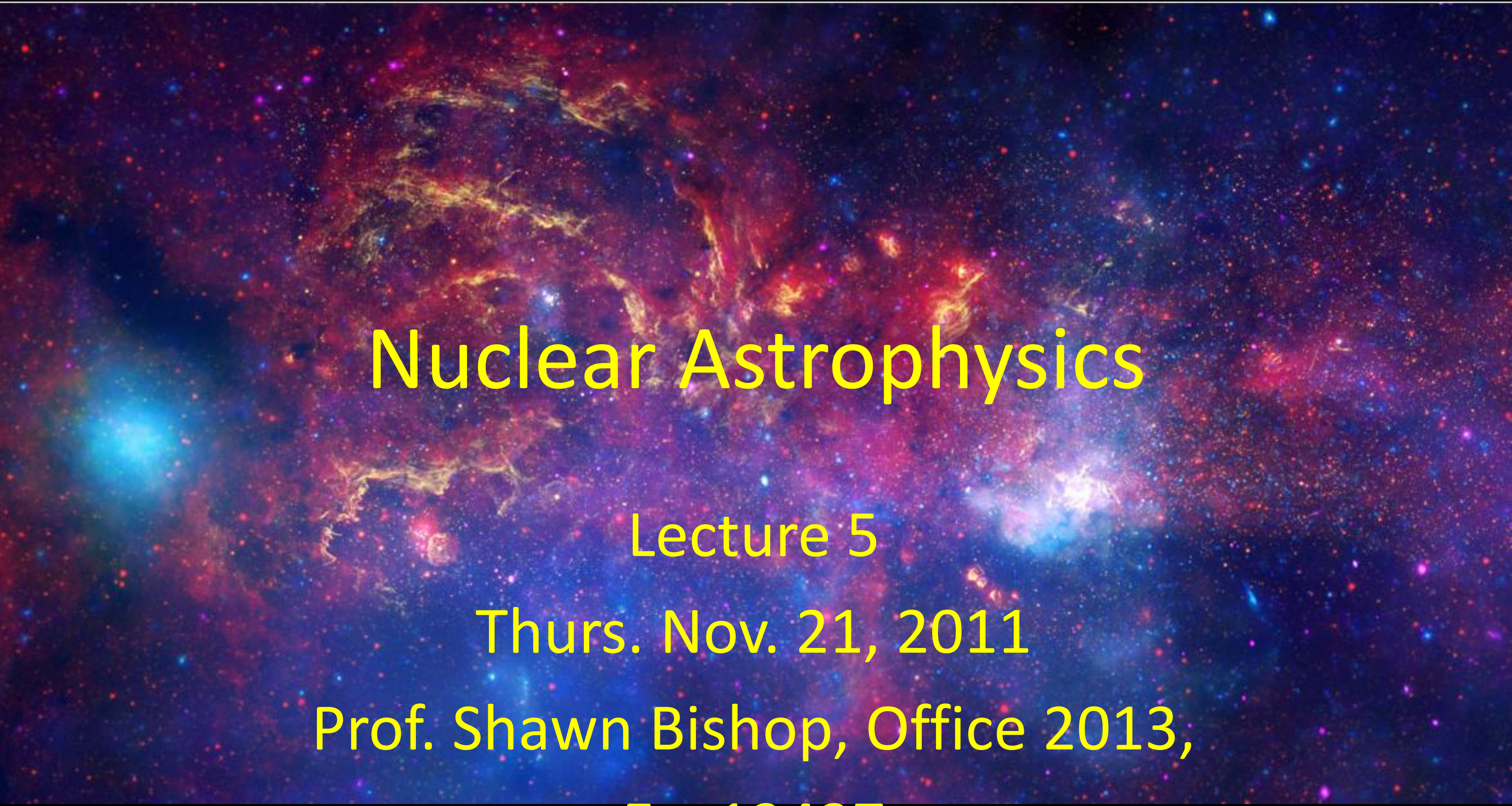

Nuclear Astrophysics

Lecture 5

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The 4 Equations of Stellar Structure

$$(A) \quad \frac{dP}{dr} = -G\frac{M(r)}{r^2}\rho(r)$$

$$(B) \quad \frac{dM(r)}{dr} = 4\pi r^2 \rho$$

$$(C) \quad \frac{dL}{dr} = 4\pi\epsilon r^2 \rho$$

$\epsilon =$ Energy generation rate per unit mass of material

$$(D) \quad L(r) = -4\pi r^2 \frac{c}{\rho\bar{\kappa}}\frac{dP_\gamma}{dr}$$

$\bar{\kappa} =$ average opacity coefficient in the material

Ancillary Equations: Pressures

$$P_{gas} = n\tau \qquad , n = N/V \qquad \tau \equiv kT$$

$$P_e = -\frac{\partial U}{\partial V} = \frac{\pi^3}{15m}\hbar^2 \left(\frac{3n_e}{\pi}\right)^{5/3}$$

$$P_e^{rel} = \frac{\hbar c \pi^{2/3}}{12}(3n_e)^{4/3}$$

$$n_e \equiv N_e/V$$

$$P_\gamma = \frac{\pi^2}{45\hbar^3 c^3}\tau^4 = \frac{1}{3}\frac{U_\gamma}{V} = \frac{1}{3}u$$

$U_\gamma =$ Internal energy of photon gas

Generalized Adiabatic Coefficients

First, let's go back to the **First Law** of Thermodynamics and something already familiar:

$$dQ = dU + PdV$$

Take the internal energy to be functions of T and V: $U = U(T, V)$

Then, by definition: $\Rightarrow dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV$

$$= \left(\frac{\partial U}{\partial \tau}\right)\left(\frac{\partial \tau}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV = \left(\frac{\partial U}{\partial \tau}\right)_V d\tau + \left(\frac{\partial U}{\partial V}\right)_T dV$$

$$dQ = \left(\frac{\partial U}{\partial \tau}\right)_V d\tau + \left(\frac{\partial U}{\partial V}\right)_T dV + PdV$$

For an ideal gas: $U = \frac{3}{2}N\tau$ and $PV = N\tau$

$$dQ = \left(\frac{\partial U}{\partial \tau}\right)_V d\tau + \left[P + \left(\frac{\partial U}{\partial V}\right)_T\right] dV$$

So, we have: $\partial U/\partial \tau = \frac{3}{2}N$ and: $PdV + VdP = Nd\tau$

Heat Capacity at constant volume: $c_V \equiv \left(\frac{\partial Q}{\partial \tau}\right)_V = \left(\frac{\partial U}{\partial \tau}\right)_V = \frac{3}{2}N$

Heat Capacity at constant pressure: $c_P \equiv \left(\frac{\partial Q}{\partial \tau}\right)_P$

$$= \frac{3}{2}N + N = \frac{5}{2}N$$

$PdV + VdP = Nd\tau$

When dP = 0 $\Rightarrow PdV = Nd\tau$

Summarizing: $c_V = \frac{3}{2}N$

$c_P = \frac{5}{2}N$

Ideal Gas adiabatic exponent:

$$\gamma \equiv c_P/c_V = \frac{5}{3}$$

Let's go back to first law, now, for ideal gas:

$$dQ = dU + PdV = \left(\frac{\partial U}{\partial \tau}\right) d\tau + PdV$$

$$= c_V d\tau + N\tau \frac{dV}{V}$$

using $U = \frac{3}{2}N\tau$ and $PV = N\tau$

$$= c_V d\tau + (c_P - c_V)\tau \frac{dV}{V}$$

$$c_V = \frac{3}{2}N \quad c_P = \frac{5}{2}N$$

For an *adiabatic* change in the gas, dQ = 0 $\Rightarrow$

$$\boxed{\frac{d\tau}{\tau} + (\gamma - 1)\frac{dV}{V} = 0}$$

From EOS we have: $\frac{dV}{V} + \frac{dP}{P} = \frac{d\tau}{\tau}$. Use this above to also get two more:

$$\boxed{\gamma \frac{dV}{V} + \frac{dP}{P} = 0}$$

$$\boxed{\gamma \frac{d\tau}{\tau} + (1 - \gamma)\frac{dP}{P} = 0}$$

When integrated, these 3 equations lead to the familiar adiabatic formulae for an ideal gas:

$$\tau V^{\gamma-1} = \text{const1} \quad \tau^{\gamma} P^{1-\gamma} = \text{const2} \quad PV^{\gamma} = \text{const3}$$

Mixture of Ideal and Photon Gases

On page 4 we had the general result:

$$dQ = \left(\frac{\partial U}{\partial \tau}\right)_V d\tau + \left(\frac{\partial U}{\partial V}\right)_T dV + PdV$$

The total pressure of the gas:

$$P = P_g + P_\gamma = \frac{1}{3}a\tau^4 + \frac{N}{V}\tau$$

$$a = \frac{\pi^2}{15\hbar^3 c^3}$$

The total internal energy of the gas:

$$U = \frac{3}{2}N\tau + aV\tau^4$$

$$P_\gamma = \frac{\pi^2}{45\hbar^3 c^3}\tau^4 = \frac{1}{3}\frac{U_\gamma}{V}$$

Start doing the partial derivatives:

$$\left(\frac{\partial U}{\partial V}\right)_T = a\tau^4 = 3P_\gamma$$

$$\left(\frac{\partial U}{\partial \tau}\right)_V = \frac{3}{2}N + 4aV\tau^3 = \frac{V}{\tau}\left(c_V\frac{\tau}{V} + 4a\tau^4\right)$$

$$= \frac{V}{\tau}\left(c_V\frac{P_g}{N} + 12P_\gamma\right)$$

But, remember that: $N = c_P - c_V$

So, we have: $$\left(\frac{\partial U}{\partial \tau}\right)_V = \frac{V}{\tau}\left(\frac{1}{\gamma - 1}P_g + 12P_\gamma\right)$$ $\gamma \equiv c_P/c_V$

And finally: $$dQ = \frac{V}{\tau}\left(\frac{1}{\gamma - 1}P_g + 12P_\gamma\right)d\tau + (4P_\gamma + P_g)\,dV$$

For adiabatic changes to the gas, we require dQ = 0. With this condition in the previous equation, we have:

$$\left(\frac{1}{\gamma-1}P_g + 12P_\gamma\right)\frac{d\tau}{\tau} + (4P_\gamma + P_g)\frac{dV}{V} = 0 \qquad (*)$$

This equation (*) is central to what follows, so “get to know” it well!

Back on page 6, for the case of an Ideal Gas only, we found 3 differential equations that related the adiabatic exponent γ to the temperature, volume and pressure of the gas. Here, we have a mixed gas of particles and photons, but let us use the ***structure*** of the equations on page 6 as a guide for building analogous equations for the case of a photon + ideal gas. Here they are:

$$\frac{dP}{P} + \Gamma_1\frac{dV}{V} = 0 \qquad (1)$$

$$\frac{dP}{P} + \frac{\Gamma_2}{1-\Gamma_2}\frac{d\tau}{\tau} = 0 \qquad (2)$$

$$\frac{d\tau}{\tau} + (\Gamma_3 - 1)\frac{dV}{V} = 0 \qquad (3)$$

The adiabatic exponents $\Gamma_1, \; \Gamma_2, \; \Gamma_3$ need to be determined. Let's proceed.

Go back to equation (*) on page 10. Compare it directly to equation (3) of page 10.

$$\left(\frac{1}{\gamma - 1} P_g + 12 P_\gamma \right) \frac{d\tau}{\tau} + (4P_\gamma + P_g) \frac{dV}{V} = 0 \qquad (*)$$

$$\frac{d\tau}{\tau} + (\Gamma_3 - 1) \frac{dV}{V} = 0 \qquad (3)$$

Thus, we have the following result: $\Gamma_3 - 1 = \dfrac{4P_\gamma + P_g}{12 P_\gamma + \frac{1}{\gamma - 1} P_g}$

In the gas, some fraction of the total pressure is from the Ideal part. Call it $P_g = \beta P$.
Photon part carries a fraction $P_\gamma = (1 - \beta) P$ of the total pressure.

Substitute these in above to get:

$$\boxed{\Gamma_3 - 1 = \frac{(4 - 3\beta)(\gamma - 1)}{\beta + 12(1 - \beta)(\gamma - 1)}}$$

Now to get Γ_1 .

Now to equation (1) of page 10: $\frac{dP}{P} + \Gamma_1 \frac{dV}{V} = 0 \qquad (1)$

First, we have the total gas pressure: $P = P_g + P_\gamma = \frac{1}{3} a\tau^4 + \frac{N}{V}\tau$

$$\Rightarrow \quad dP = 4 \cdot \frac{1}{3} a\tau^3 d\tau + \frac{N}{V} d\tau - \frac{N}{V^2}\tau$$

$$= (4P_\gamma + P_g)\frac{d\tau}{\tau} - P_g \frac{dV}{V} \qquad (**)$$

Substitute this into equation (1):

$$\frac{1}{P}\left[(4P_\gamma + P_g)\frac{d\tau}{\tau} - P_g \frac{dV}{V}\right] + \Gamma_1 \frac{dV}{V} = 0$$

$$\frac{1}{P}(4P_\gamma + P_g)\frac{d\tau}{\tau} + \left(\Gamma_1 - \frac{P_g}{P}\right)\frac{dV}{V} = 0$$

Using the relationships: $P_g = \beta P$ and $P_\gamma = (1-\beta)P$, we have:

$$[4(1-\beta)+\beta]\,\frac{d\tau}{\tau} + (\Gamma_1 - \beta)\,\frac{dV}{V} = 0 \qquad (***)$$

Now, we know from equation (3) the adiabatic exponent Γ_3 (solved it on page 11).
And, above, we can replace dτ/τ using equation (3).

$$\frac{d\tau}{\tau} + (\Gamma_3 - 1)\frac{dV}{V} = 0$$

Homework: Substitute into (* * *) for dτ/τ and use the result for $\Gamma_3 - 1$ on page 11 to determine the formula for Γ_1 .

Result:

$$\Gamma_1 = \beta + \frac{(4-3\beta)^2(\gamma-1)}{\beta + 12(1-\beta)(\gamma-1)}$$

We now have $\Gamma_1,\ \Gamma_3$, and we still need Γ_2. There are different ways to get it. One way is this:

Take the difference between equations (1) and (2) and add that result to equation (3). Should get:

$$\Gamma_1 \frac{dV}{V} + \frac{\Gamma_2}{1-\Gamma_2}(\Gamma_3 - 1)\frac{dV}{V} = 0$$

$$\Rightarrow \Gamma_1 + \frac{\Gamma_2}{1-\Gamma_2}(\Gamma_3 - 1) = 0$$

$$\Rightarrow \Gamma_2 = -\frac{\Gamma_1}{(\Gamma_3 - 1) - \Gamma_1}$$

You've got Γ_1 and $\Gamma_3 - 1$. The remaining algebra is for **you** to do. Show:

$$\Gamma_2 = \frac{\beta^2 + (\gamma - 1)(16 - 12\beta - 3\beta^2)}{\beta^2 + 3(\gamma - 1)(4 - 3\beta - \beta^2)}$$

Summarizing Adiabatic Exponents

$$\frac{dP}{P} + \Gamma_1 \frac{dV}{V} = 0 \qquad \Gamma_1 = \beta + \frac{(4-3\beta)^2(\gamma-1)}{\beta + 12(1-\beta)(\gamma-1)}$$

$$\frac{dP}{P} + \frac{\Gamma_2}{1-\Gamma_2}\frac{d\tau}{\tau} = 0 \qquad \Gamma_2 = \frac{\beta^2 + (\gamma-1)(16-12\beta-3\beta^2)}{\beta^2 + 3(\gamma-1)(4-3\beta-\beta^2)}$$

$$\frac{d\tau}{\tau} + (\Gamma_3 - 1)\frac{dV}{V} = 0 \qquad \Gamma_3 - 1 = \frac{(4-3\beta)(\gamma-1)}{\beta + 12(1-\beta)(\gamma-1)}$$

Some Theorems of Stellar Equilibrium

Let's take two of the 4 stellar structure equations and run with them; see where they take us.

$$\frac{dP}{dr} = \frac{d}{dr}(P_g + P_\gamma) = -G\frac{M(r)}{r^2}\rho(r)$$

Hydrostatic equilibrium

$$\frac{dP_\gamma}{dr} = -\frac{\kappa L(r)}{4\pi c r^2}\rho$$

Radiative Transport. (Equation (D) on page 2, rearranged)

Dividing the 2nd by the 1st, leaves us:

$$\frac{dP_\gamma}{dP} = \frac{\kappa}{4\pi c G M(r)} L(r)$$

Remember: $\kappa = \bar{\kappa}$, which is the opacity averaged over **all** photon frequencies.

Next, let's define the following quantity: $\eta(r) \equiv \dfrac{\left(\frac{L(r)}{M(r)}\right)}{\left(\frac{L_*}{M_*}\right)} = \dfrac{\bar{\epsilon}(r)}{\bar{\epsilon}}$

This is just the average energy rate per unit mass interior to the point "r" divided by the total energy generation rate per unit mass of the entire star.

$$\frac{L_*}{M_*} \equiv \frac{L(R)}{M(R)}$$

By its definition, $\eta \geq 1$

We can now write:

$$\frac{dP_\gamma}{dP} = \frac{\kappa}{4\pi c G M(r)} L(r)$$

$$= \frac{L_*}{4\pi c G M_*} \kappa \eta(r)$$

Nothing has changed here; it's all still **exact**. All we have done is buried the dependence on radial coordinate, r, into the $\kappa\eta$ term.

But, now we can derive a simple theorem, that limits the value of the stellar opacity at **all points in the star,** from this result. Let's see.

We have: $\frac{dP_\gamma}{dP} = \frac{L_*}{4\pi c G M_*}\kappa_r \eta(r)$ and, also: $P = P_g + P_\gamma$

Also, $dP = dP_g + dP_\gamma \equiv a_g + b_\gamma$

$$\frac{dP_\gamma}{dP} = \frac{b_\gamma}{a_g + b_\gamma} = \frac{1}{1 + a_g/b_\gamma} < 1$$

Therefore, we have: $\frac{dP_\gamma}{dP} = \frac{L_*}{4\pi c G M_*}\kappa_r \eta(r) < 1$

Also, because $\eta \geq 1$ $\frac{L_*}{4\pi c G M_*}\kappa_r \leq \frac{L_*}{4\pi c G M_*}\kappa_r \eta(r) < 1$

Finally: $\frac{L_*}{4\pi c G M_*}\kappa_r < 1$

Or, a limit on opacity: $\kappa_r < \frac{4\pi c G M_*}{L_*}$

Let's return back to: $\frac{dP_\gamma}{dP} = \frac{L_*}{4\pi c G M_*}\kappa\eta(r)$. And let's integrate it.

We have: $\Rightarrow \int_r^R dP_\gamma = P_\gamma(R) - P_\gamma(r) = \frac{L_*}{4\pi c G M_*}\int_P^0 \kappa\eta(r)dP$

$P_\gamma(R) = 0$

So, after reversing the integral limits, and dividing by unity [P(r)/P(r)], we have:

$$P_\gamma(r) = \frac{L_* P(r)}{4\pi c G M_*}\frac{1}{P}\int_0^P \kappa\eta(r)dP$$

Pressure-averaged quantity. Call it: $\overline{\kappa\eta}$

So, in terms of the total mass of the star, and its total luminosity, we can write:

$P_\gamma(r) = \frac{L_* P(r)}{4\pi c G M_*}\overline{\kappa\eta}$. Now use: $P_\gamma = (1-\beta)P$

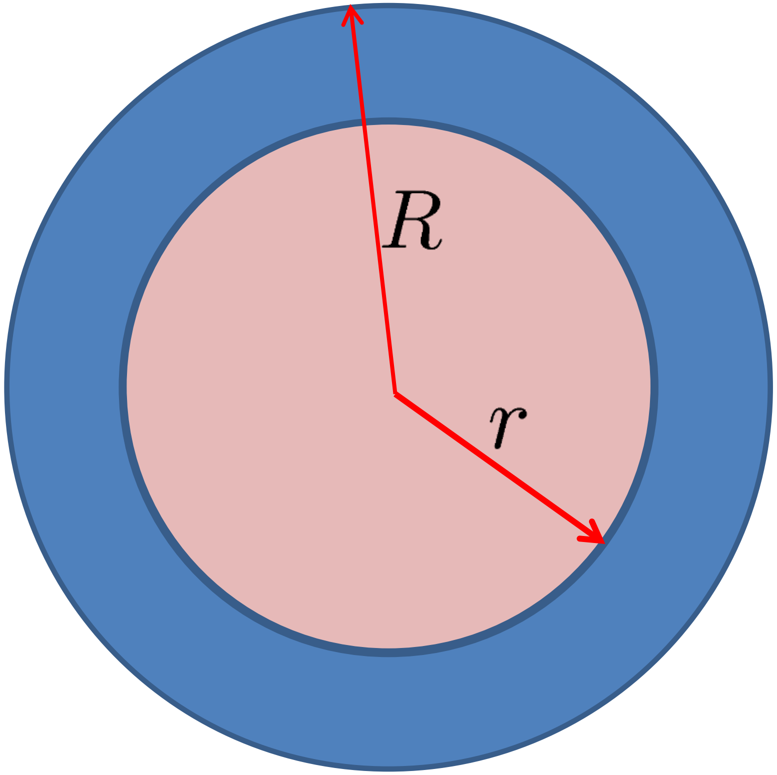
And now we can write:

$$\overline{\kappa\eta} = \frac{4\pi c G M_*(1-\beta)}{L_*}$$

The previous result is **Strömgren's Theorem**. In words, we have:

"The ratio of the radiation pressure to the total pressure at a point inside a star in radiative equilibrium is proportional to the **average value** of $\kappa\eta$ for the regions **exterior** to the point r, with the average being taken with respect to dP."

Explaining this with a diagram (picture worth 1000 words): The ratio of radiation pressure to total pressure on surface of pink sphere (radius r) is proportional to the pressure average of $\kappa\eta$ in the region of the blue shell of radius R-r.



The Concept of Convection

We can use the Strömgren Theorem equation to write $\frac{dP_\gamma}{dP} = \frac{L_*}{4\pi c G M_*}\kappa\eta(r)$ as:

$$\frac{dP_\gamma}{dP} = (1-\beta)\frac{\kappa\eta(r)}{\overline{\kappa\eta}(r)}$$

But, we have: $\frac{P_\gamma}{P} = (1-\beta)$, so we can write: $\frac{dP_\gamma}{P_\gamma} = \frac{\kappa\eta(r)}{\overline{\kappa\eta}(r)}\frac{dP}{P}$ $(* * **)$

Radiation pressure: $P_\gamma = \frac{1}{3}a\tau^4$

$$\Rightarrow dP_\gamma = 4\cdot\frac{a}{3}\tau^3 d\tau = 4P_\gamma\frac{d\tau}{\tau}$$

Sub this into (* * * *):

$$\boxed{\frac{d\tau}{\tau} = \frac{1}{4}\frac{\kappa\eta(r)}{\overline{\kappa\eta}(r)}\frac{dP}{P}}$$

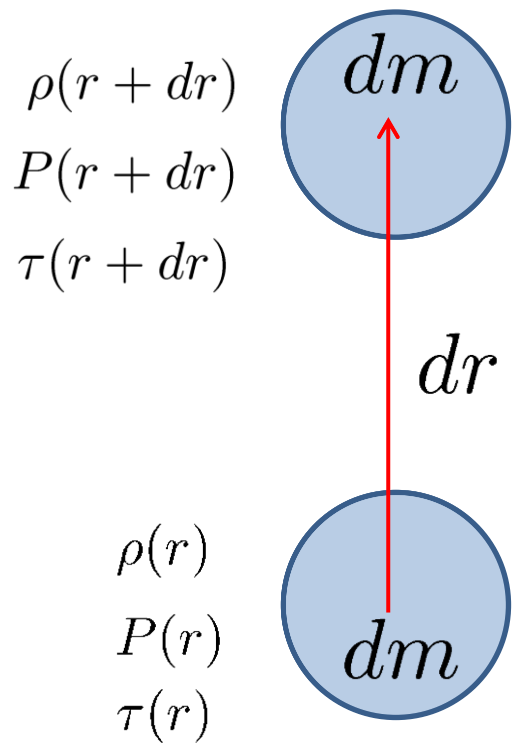
Compare with Adiabatic Exponent equation 2, on pages 10 and 15!

We just derived: $$\frac{d\tau}{\tau} = \frac{1}{4}\frac{\kappa\eta(r)}{\overline{\kappa\eta}(r)}\frac{dP}{P}$$

And 2nd Adiabatic Equation (rearranged): $$\frac{d\tau}{\tau} = \frac{\Gamma_2 - 1}{\Gamma_2}\frac{dP}{P}$$

Consider a mass element dm. Suppose it undergoes an increase in temperature relative to its surroundings. The temperature increase will cause dm to expand, and its density to become less than the surroundings.

It will, therefore, make a displacement to a region of lower density.



Assume:

1. Pressure exerted by dm on surroundings is equal to pressure surroundings exert on dm.
2. Expansion (or contraction) of dm occurs **adiabatically**.
3. Friction can be neglected.

By the adiabatic assumption: $$\left(\frac{d\tau}{\tau}\right)_{dm} = \frac{\Gamma_2 - 1}{\Gamma_2}\frac{dP}{P}$$

And Γ_2 is what we (or what you will) derived on page 14.

$$\Gamma_2 = \frac{\beta^2 + (\gamma - 1)(16 - 12\beta - 3\beta^2)}{\beta^2 + 3(\gamma - 1)(4 - 3\beta - \beta^2)}$$

By the first assumption, the dP/P terms will be the same for the surroundings and for dm.

$$\left(\frac{d\tau}{\tau}\right)_{dm} = \frac{\Gamma_2 - 1}{\Gamma_2}\frac{dP}{P} \qquad \frac{d\tau}{\tau} = \frac{1}{4}\frac{\kappa\eta(r)}{\overline{\kappa\eta}(r)}\frac{dP}{P}$$

This means: the temperature change of dm is **different** from the change in temperature of the surroundings.

If $\frac{\Gamma_2 - 1}{\Gamma_2} > \frac{1}{4}\frac{\kappa\eta}{\overline{\kappa\eta}(r)}$, then dm moves outward until it has the same temp. and density as the surroundings. The converse means dm will sink until its temperature and density are the same as surroundings.

Thus, if the mass bubble, dm, has internal conditions such that $\frac{\Gamma_2 - 1}{\Gamma_2} > \frac{1}{4}\frac{\kappa\eta}{\overline{\kappa\eta}(r)}$ then the star will be convective; the energy is carried away by convection, which is the movement of hot material to regions of lower temperature. When the bubble and surroundings have the same temperature, at that point, the star carries the energy by by **radiative** transport.

When convective transport is dominant, then we have:

$$\frac{d\tau}{\tau} = \frac{\Gamma_2 - 1}{\Gamma_2}\frac{dP}{P}$$

Now, we have: $d\tau = \frac{d\tau}{dr} \cdot dr$ and: $dP = \frac{dP}{dr} \cdot dr$

$$\frac{d\tau}{dr} = \frac{\Gamma_2 - 1}{\Gamma_2}\frac{\tau}{P}\frac{dP}{dr}$$