Nuclear Astrophysics

Lecture 4

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ϵ

r

$r + \Delta r$

Energy generation from nuclear reactions is a function of temperature, density and the set of composition parameters $\{X_z\}$. So $\epsilon = \epsilon(T, \rho, \{X_z\})$, and is the energy rate per unit mass of stellar material.

Let $L(r)$ be the energy rate flowing outward through a spherical surface of radius r.

Energy generated in dm + energy entering dm = energy flowing out of dm. Or,

$$L(r + \Delta r) - L(r) = \epsilon(r)\rho(r)4\pi r^2 \Delta r$$

$$\Rightarrow \boxed{\frac{dL}{dr} = 4\pi\epsilon\rho r^2}$$

This is the 3rd stellar structure equation for a **static** star. We have two more to derive. It can also be expressed as a mass derivative quite simply:

$$\frac{dL}{dm} = \epsilon$$

THE ROAD TO ENERGY TRANSPORT IN STARS

We have just derived the relation between the luminosity gradient in a star. It depends on the local energy generation rate and the local density of the material This, in turn, is connected to the nuclear reactions occurring in that material.

Before we can get to nuclear reaction physics, however, we require one more stellar structure equation.

It is clear that the energy generation rate in the star, being **density** dependent (and temperature dependent because nuclear reaction rates are **highly** sensitive to temperature), must produce a **temperature gradient** in the star. We know this, of course: the center of the star is hottest and the surface is where the energy escapes to space. Heat flows from hot to cold, so the star has a temperature gradient.

This temperature gradient is responsible for the transport of heat to the surface. The carriers of the thermal energy, for a star consisting of a mixture of ideal gas and radiation (like Main Sequence stars), are photons.

Our final stellar structure equation must somehow connect the luminosity of the star with the temperature gradient. Our system of stellar structure equations will, then, be closed.

Reviewing your Lecture 2 notes, you will find on page 18 (after some algebra), that the energy density of a photon gas is:

$$u \equiv \frac{U}{V} = \frac{\hbar}{\pi^2 c^3} \int_0^\infty \frac{\omega^3}{\exp(\hbar\omega/\tau) - 1} d\omega$$

This suggests defining a spectral energy density (energy per unit volume per unit frequency interval) as:

$$u_\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{\exp(\hbar\omega/\tau) - 1}$$

We know also, from our work in Lecture 2, that the **total** photon gas pressure is:

(*) $$P_\gamma = \frac{1}{3} \int_0^\infty u_\omega d\omega = \frac{1}{3} a T^4 \qquad\qquad a = \frac{\pi^2 k^4}{15 \hbar^3 c^3}$$

This result further suggests a definition for the "spectral pressure", which can be thought of as the fractional contribution to the total pressure by photons of angular frequency ω:

$$p_\omega = \frac{1}{3} u_\omega$$

Using the last expression and Equation (*) we can obviously write:

$$\frac{\partial P_\gamma}{\partial T} \equiv \int_0^\infty \frac{\partial p_\omega}{\partial T} d\omega = \frac{4}{3} a T^3$$
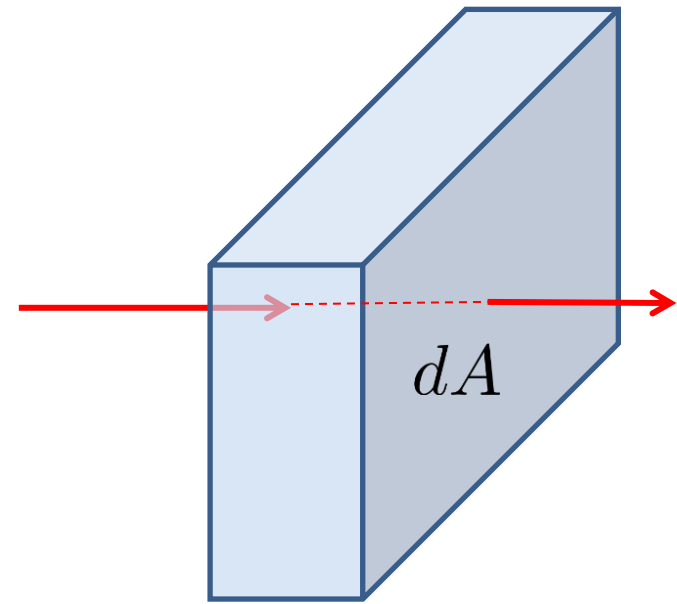
From which, it trivially follows:

(**) $$\frac{4}{3} a T^3 \bigg/ \int_0^\infty \frac{\partial p_\omega}{\partial T} d\omega = 1$$

Remember this last result. With it, we can now derive the stellar temperature gradient in terms of the photon luminosity.

Difference in radiation force f_ω across the slab:

$$\Delta f_\omega = p_\omega(x + \Delta x)dA - p_\omega(x)dA$$

$$= [p_\omega(x + \Delta x) - p_\omega(x)]dA$$

dA

Furthermore, a photon **flux** F_ω, when passing through a thin slab of material, will suffer an attenuation in flux. The change in flux of this beam, for normal incidence (as is the case with our geometry) is given by:

$$\Delta F_\omega = -\kappa_\omega \rho(x) F_\omega(x) \Delta x$$

Now, F_ω is the energy per frequency interval carried by all photons with frequency crossing through dA per unit time.

Dividing it by c will yield the momentum flux (total momentum) carried by all photons with angular frequency ω crossing through dA per unit time.

Thus: $$\Delta f_\omega = \frac{\Delta F_\omega}{c}dA = -\frac{\kappa_\omega \rho(x)}{c} F_\omega(x) \Delta x dA$$

Equate the bottom expression to what we have at the top of previous slide, and take the limit as $\Delta r \to 0$:

$$\frac{dp_\omega(x)}{dx} = -\frac{\kappa_\omega \rho(x)}{c} F_\omega(x)$$

$$\Rightarrow F_\omega(x) = -\frac{c}{\kappa_\omega \rho(x)} \frac{dp_\omega(x)}{dx}$$

$$\Rightarrow F_\omega(x) = -\frac{c}{\kappa_\omega \rho(x)} \frac{dT}{dx} \frac{\partial p_\omega(x)}{\partial T}$$

Now we must integrate over all photon frequencies. LHS is just, $F(x)$ (by definition)

(***) $$\Rightarrow F(x) = -\frac{c}{\rho(x)} \frac{dT}{dx} \int_0^\infty \frac{1}{\kappa_\omega} \frac{\partial p_\omega}{\partial T} d\omega$$

We are almost done: Remember the identity (**) on page 6. Multiply (***) by it. We have:

$$\Rightarrow \quad F(r) = -\frac{c}{\rho(r)}\frac{4}{3}aT^3\frac{dT}{dr}\underbrace{\int_0^\infty \frac{1}{\kappa_\omega}\frac{\partial p_\omega}{\partial T}d_{,\omega}\bigg/\int_0^\infty \frac{\partial p_\omega}{\partial T}d\omega}$$

Averaged κ over distribution of $\frac{\partial u_\nu}{\partial T}$. κ is the average opacity

Finally, the **luminosity** of our wedge, once integrated over a spherical shell, is just: $L(r) = F(r) \times 4\pi r^2$. We have, therefore, that:

$$\Rightarrow \quad L(r) = -\frac{4\pi c r^2}{\rho(r)\bar{\kappa}}\frac{4}{3}aT^3\frac{dT}{dr}$$

Or, more compact:

$$L(r) = -\frac{4\pi c r^2}{\rho(r)\bar{\kappa}}\frac{dP_\gamma}{dr}$$

Recall from lecture 2 the condition for hydrostatic equilibrium:

$$\frac{dP_{tot}}{dr} = -G\frac{M(r)}{r^2}\rho(r)$$

Also recall for a polytrope star with particle and radiation pressure the definitions of lecture 3

$$P_\gamma = (1 - \beta)P_{tot}$$

Use the above two expressions to get dP_{tot}/dr and substitute into $L(r)$ from previous page

$$L(r) = 4\pi\frac{c}{\kappa}(1 - \beta)GM(r)$$

For massive stars, Thomson scattering dominates the opacity (Compton scattering on free electrons). The cross section for this process is a constant. For complete ionization it leads to a opacity of 0.4 cm²/g

Eddington's Quartic Equation: we derived the mass of a radiation + particle gas star

$$M = 18.0 \frac{\sqrt{1-\beta}}{\mu^2 \beta^2} M_\odot \qquad \Rightarrow 1 - \beta \propto \beta^4 M^2$$

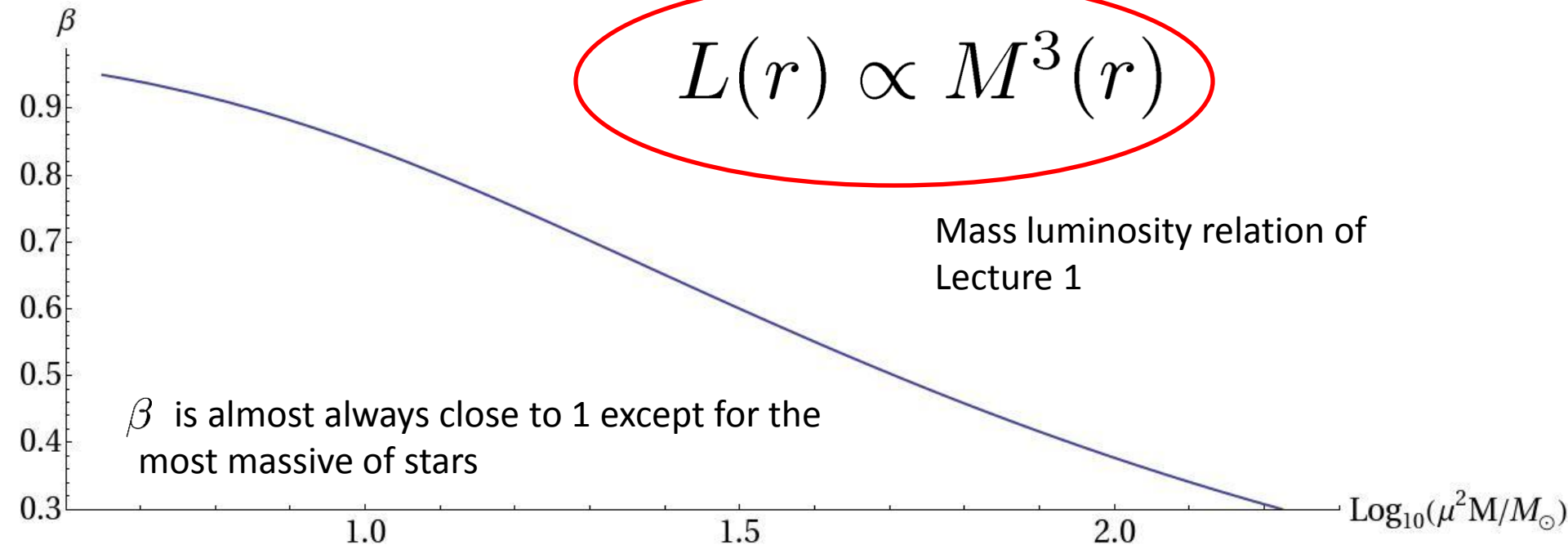Substitute this into the previous luminosity expression:

$$L(r) \propto \beta^4 M^3(r)$$

Assuming β is constant over entire star, then

$$L(r) \propto M^3(r)$$

Mass luminosity relation of Lecture 1

β is almost always close to 1 except for the most massive of stars

Sources of Opacity

• Bound-Bound Absorption: Absorption of a photon by an atom, causing an upward transition to electron orbital of higher energy. It is a true-absorption process; its inverse is normal emission via downward transitions.

• Bound-Free Absorption: Absorption of a photon by an atom causing a bound electron to make a transition to the continuum. True-absorption process; inverse is radiative recombination.

• Free-Free Absorption: Absorption of a photon by a **continuum** electron as is passes an ion and makes a transition to another continuum state at higher energy. A true-absorption process; inverse is *bremsstrahlung*.

• Scattering from free electrons: Scattering of photons by individual free electrons in the gas, and known as Compton Scattering; in non-relativistic limit, called Thomson scattering. Not true-absorption as the photon energy remains unchanged.
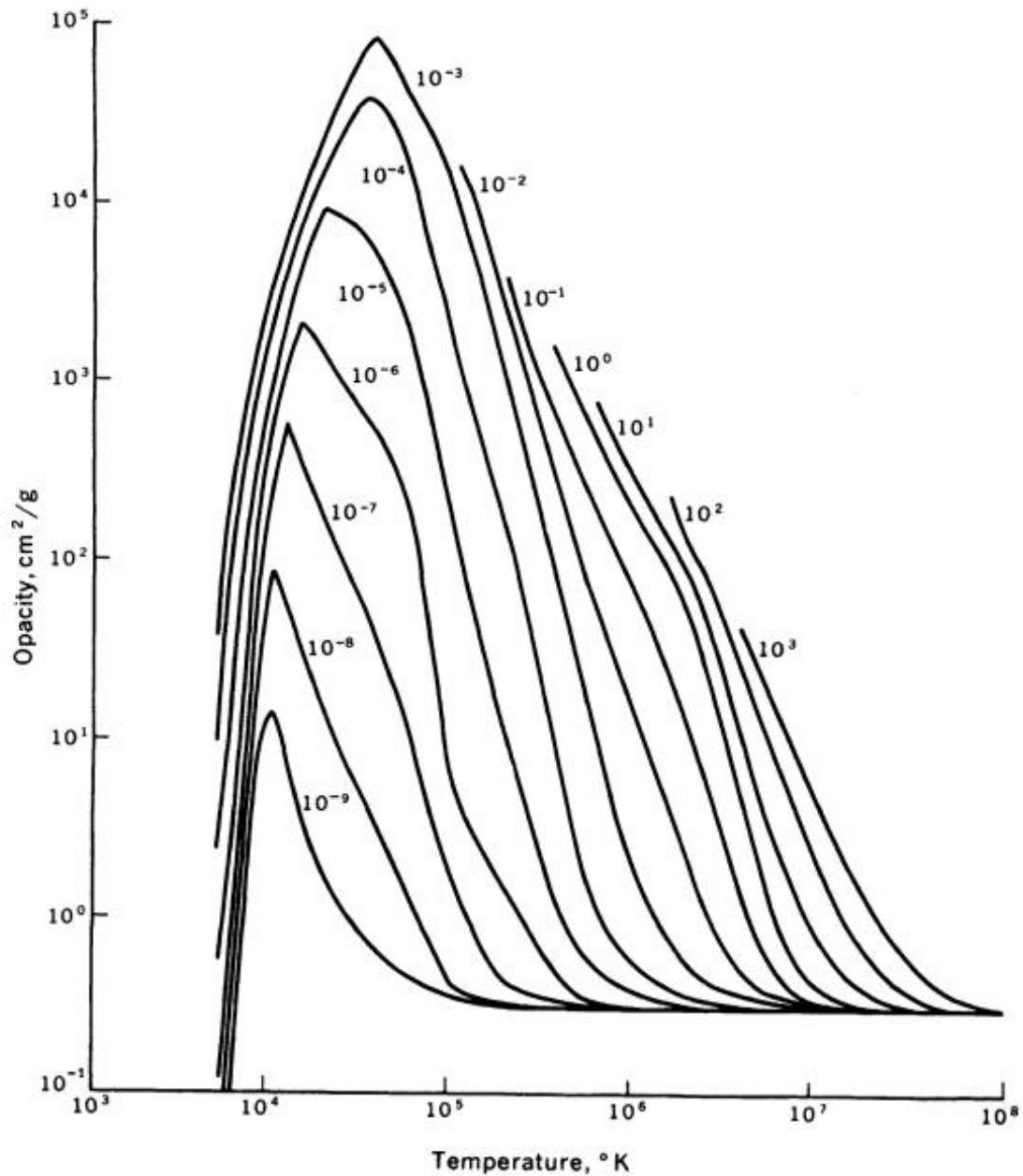
Fig. 3-16 The total opacity of material of solar composition as a function of temperature. Each curve is labeled by the value of the density. The range of values

The 4 Equations of Stellar Structure

(A) $$\frac{dP}{dr} = -G\frac{M(r)}{r^2}\rho(r)$$

(B) $$\frac{dM(r)}{dr} = 4\pi r^2 \rho$$

(C) $$\frac{dL}{dr} = 4\pi \epsilon r^2 \rho$$ $\qquad$ $\epsilon =$ Energy generation rate per unit mass of material

(D) $$L(r) = -4\pi r^2 \frac{c}{\rho\bar{\kappa}}\frac{dP_\gamma}{dr}$$ $\qquad$ $\bar{\kappa} =$ average opacity coefficient in the material

Ancillary Equations: Equations of State

$$P_{gas} = n\tau \qquad , n = N/V \qquad \tau \equiv kT$$

$$P_e = -\frac{\partial U}{\partial V} = \frac{\pi^3}{15m}\hbar^2\left(\frac{3n_e}{\pi}\right)^{5/3}$$

$$P_e^{rel} = \frac{\hbar c \pi^{2/3}}{12}(3n_e)^{4/3}$$

$$n_e \equiv N_e/V$$

$$P_\gamma = \frac{\pi^2}{45\hbar^3 c^3}\tau^4 = \frac{1}{3}\frac{U_\gamma}{V} = \frac{1}{3}u \qquad U_\gamma = \text{Internal energy of photon gas}$$

Generalized Adiabatic Coefficients

First, let's go back to the **First Law** of Thermodynamics and something already familiar:

$$dQ = dU + PdV$$

Take the internal energy to be functions of T and V: $U = U(T, V)$

Then, by definition: $\Rightarrow dU = \left(\dfrac{\partial U}{\partial T}\right)_V dT + \left(\dfrac{\partial U}{\partial V}\right)_T dV$

$$= \left(\dfrac{\partial U}{\partial \tau}\right)\left(\dfrac{\partial \tau}{\partial T}\right)_V dT + \left(\dfrac{\partial U}{\partial V}\right)_T dV = \left(\dfrac{\partial U}{\partial \tau}\right)_V d\tau + \left(\dfrac{\partial U}{\partial V}\right)_T dV$$

$$\boxed{dQ = \left(\dfrac{\partial U}{\partial \tau}\right)_V d\tau + \left(\dfrac{\partial U}{\partial V}\right)_T dV + PdV}$$

For an ideal gas: $U = \frac{3}{2} N \tau$ and $PV = N\tau$

$$dQ = \left(\frac{\partial U}{\partial \tau} \right)_V d\tau + \left[P + \left(\frac{\partial U}{\partial V} \right)_T \right] dV$$

So, we have: $\partial U / \partial \tau = \frac{3}{2} N$ and: $P dV + V dP = N d\tau$

Heat Capacity at constant volume: $c_V \equiv \left(\frac{\partial Q}{\partial \tau} \right)_V = \left(\frac{\partial U}{\partial \tau} \right)_V = \frac{3}{2} N$

Heat Capacity at constant pressure: $c_P \equiv \left(\frac{\partial Q}{\partial \tau} \right)_P$

$P dV + V dP = N d\tau$

When dP = 0 $\Rightarrow P dV = N d\tau$

$$= \frac{3}{2} N + N = \frac{5}{2} N$$

Summarizing: $c_V = \frac{3}{2} N$

$c_P = \frac{5}{2} N$

Ideal Gas adiabatic exponent:

$$\gamma \equiv c_P / c_V = \frac{5}{3}$$

Let's go back to first law, now, for ideal gas:

$$dQ = dU + PdV = \left(\frac{\partial U}{\partial \tau}\right) d\tau + PdV$$

$$= c_V d\tau + N\tau \frac{dV}{V}$$

using $U = \frac{3}{2}N\tau$ and $PV = N\tau$

$$= c_V d\tau + (c_P - c_V)\tau \frac{dV}{V}$$

$$c_V = \frac{3}{2}N \qquad c_P = \frac{5}{2}N$$

For an *adiabatic* change in the gas, dQ = 0 $\Rightarrow$ $$\boxed{\frac{d\tau}{\tau} + (\gamma - 1)\frac{dV}{V} = 0}$$

From EOS we have: $\dfrac{dV}{V} + \dfrac{dP}{P} = \dfrac{d\tau}{\tau}$. Use this above to also get two more:

$$\boxed{\gamma \frac{dV}{V} + \frac{dP}{P} = 0}$$

$$\boxed{\gamma \frac{d\tau}{\tau} + (1 - \gamma)\frac{dP}{P} = 0}$$

When integrated, these 3 equations lead to the familiar adiabatic formulae for an ideal gas:

$$\tau V^{\gamma-1} = \text{const1} \quad \tau^{\gamma} P^{1-\gamma} = \text{const2} \quad PV^{\gamma} = \text{const3}$$