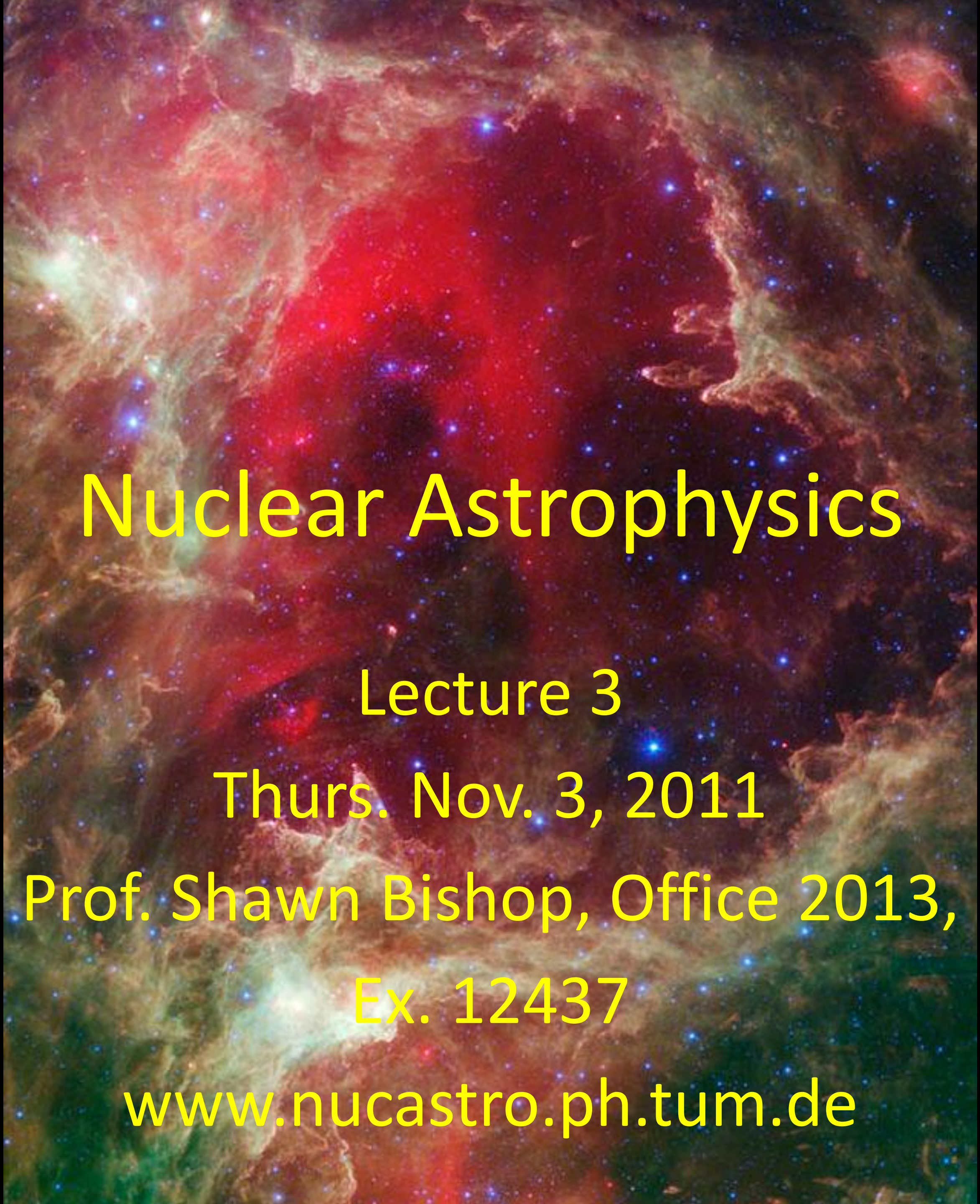

Nuclear Astrophysics

Lecture 3

Thurs. Nov. 3, 2011

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Summary of Results Thus Far

$$\frac{dP}{dr} = -G\frac{M(r)}{r^2}\rho(r)$$

$$P_{gas} = n\tau \qquad , n = N/V$$

$$P_e = -\frac{\partial U}{\partial V} = \frac{\pi^3}{15m}\hbar^2\left(\frac{3n_e}{\pi}\right)^{5/3}$$

$$n_e = N_e/V$$

$$P_e^{rel} = \frac{\hbar c \pi^{2/3}}{12}(3n_e)^{4/3}$$

$$P_\gamma = \frac{\pi^2}{45\hbar^3 c^3}\tau^4$$

Alternative expressions for Pressures

$$n = N/V = \sum_z N_z/V$$

N_z is the number of atoms of atomic species with atomic number "z" in the volume V

Mass density of each species is just: $\rho_z = \frac{N_z}{V} \times \frac{A_z}{N_A}$ where A_z and N_A are the atomic mass of species "z" and Avogadro's number, respectively

Mass fraction, in volume V, of species "z" is just $X_z = m_z/\rho V = \rho_z V/\rho V = \rho_z/\rho$

And clearly, $\sum_z X_z = 1$

Collect the algebra to write $N_z/V = \rho_z N_A/A_z = N_A \frac{X_z}{A_z}\rho$

And so we have for n:

$$n = \rho N_A \sum_z \frac{X_z}{A_z}$$

If species "z" can be ionized, the number of particles can be $N_z \rightarrow N_z n_z$
where n_z is the number of free particles produced by species "z" (nucleus + free electrons).
If fully ionized, $n_z = 1 + z$ and

$$n = \rho N_A \sum_z \frac{X_z}{A_z}(1+z)$$

The mean molecular weight is defined by the quantity: $\frac{1}{\mu} = \sum_z \frac{X_z n_z}{A_z}$

We can write it out as:

$$\mu = \left[\frac{X}{1.008} n_H + \frac{Y}{4.004} n_{He} + (1 - X - Y) \left\langle \frac{n_z}{A_z} \right\rangle \right]^{-1}$$

$\langle n_z / A_z \rangle$ is the average of n_z / A_z for atomic species Z > 2

For atomic species heavier than helium, average atomic weight is $2z + 2$ and if fully ionized, $n_z = z + 1$

Fully ionized gas: $\mu \approx \frac{1}{2X + 3Y/4 + (1 - X - Y)/2} = \frac{2}{1 + 3X + 0.5Y}$

Same game can be played for electrons:

$$n_e = \sum_z N_z (n_z - 1) = \rho N_A \sum_z \frac{X_z}{A_z} (n_z - 1)$$

Temp. vs Density Plane

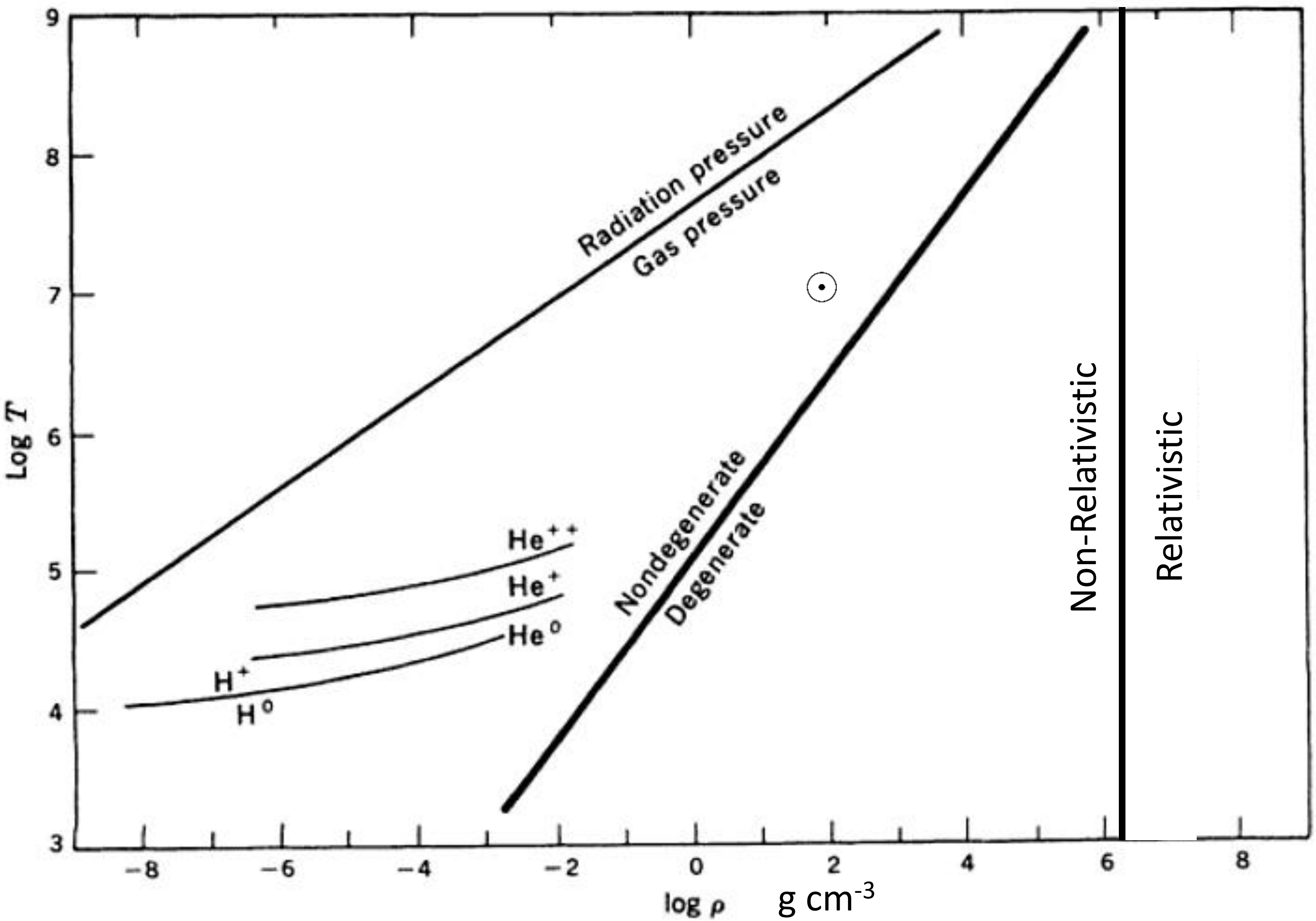


Fig. 2-11 Zones of the equation of state of a gas in thermodynamic equilibrium. Radiation

Thermodynamics of the Gas

1st Law of Thermodynamics: $dQ = dU + pdV$

dQ Thermal energy of the system (heat)

dU Total energy of the system

Assume that $U = U(T, V)$, then $dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV$

Substitute into dQ: $dQ = \left(\frac{\partial U}{\partial T}\right)_V dT + \left[P + \left(\frac{\partial U}{\partial V}\right)_T\right] dV$

Heat capacity at constant volume: $c_V = \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V$

Heat capacity at constant pressure: $c_P = \left(\frac{dQ}{dT}\right)_P = \left(\frac{\partial U}{\partial T}\right)_V + \left[P + \left(\frac{\partial U}{\partial V}\right)_T\right] \left(\frac{\partial V}{\partial T}\right)_P$

We finally have:

$$c_P - c_V = \left[P + \left(\frac{\partial U}{\partial V}\right)_T\right] \left(\frac{\partial V}{\partial T}\right)_P$$

For an ideal gas: $U = \frac{3}{2}N\tau$ and $PV = N\tau$

Therefore, $c_v = \left(\frac{\partial U}{\partial T}\right)_V = \frac{3}{2}Nk$

And, $c_P - c_V = P\left(\frac{\partial V}{\partial T}\right)_P = P \cdot N/P = Nk$

So, $c_P = \frac{5}{2}N$

Let's go back to first law, now, for ideal gas:

$$dQ = dU + pdV = \left(\frac{\partial U}{\partial \tau}\right) d\tau + PdV$$

$$= c_V d\tau + N\tau\frac{dV}{V}$$

using $U = \frac{3}{2}N\tau$ and $PV = N\tau$

For an *isentropic* change in the gas, dQ = 0

This leads to, after integration of the above with dQ = 0, and $\gamma = c_P/c_V = 5/3$

$$\tau V^{\gamma-1} = \text{const} \qquad \tau^{\gamma}P^{1-\gamma} = \text{const} \qquad PV^{\gamma} = \text{const}$$

First Law for *isentropic* changes: $dU = -PdV$

Take differentials of $\tau V^{\gamma-1} = \text{const} \quad \rightarrow \frac{d\tau}{\tau} + (\gamma - 1)\frac{dV}{V} = 0$

but $dV = -\frac{dU}{P} \quad \rightarrow \frac{d\tau}{\tau} + (1-\gamma)\frac{dU}{PV} = 0$

$PV \rightarrow N\tau$

Use $U = \frac{3}{2}N\tau = K \quad \rightarrow d\tau = \frac{2}{3N}dK$

Finally $dU = \frac{2}{3}\frac{dK}{\gamma-1}$

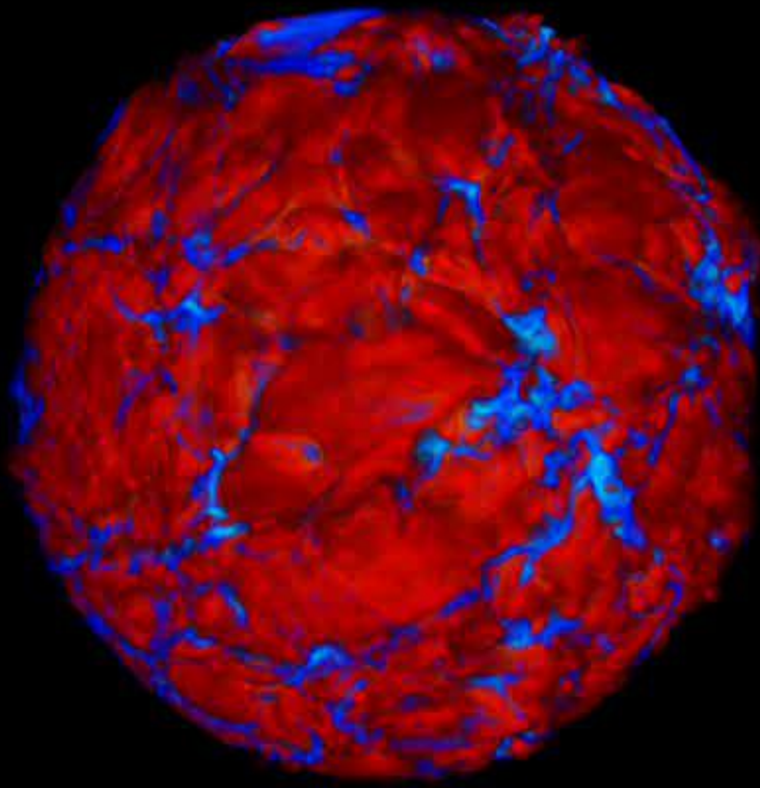
Because γ is constant, we can integrate the last equation over the star: $K = \frac{3}{2}(\gamma - 1)U$

Total energy of Star is gravitational binding energy Ω and internal energy U

$$E = U + \Omega = U - 2K = U - 3(\gamma - 1)U = -(3\gamma - 4)U$$

$\gamma > 4/3$ if star is to remain bound! (Ideal gas: $\gamma = 5/3$, so it's safe)

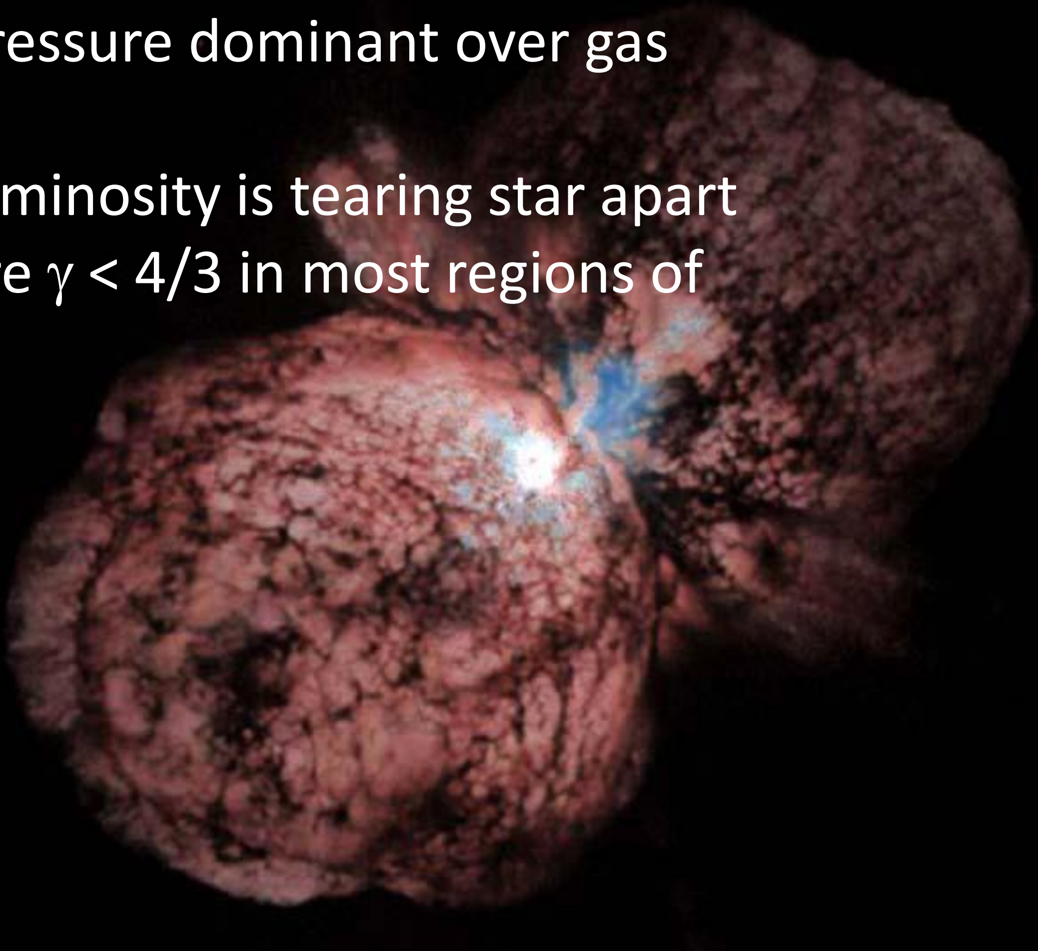
AGB Pulsation



University of Minnesota's Laboratory for Computational Science & Engineering (LCSE)

η Carinae (Southern Hemisphere)

- Eta Carinae: star of ~ 100 Solar masses
- Radiation pressure dominant over gas pressure
- Radiation luminosity is tearing star apart
- A case where $\gamma < 4/3$ in most regions of the star!



Hubble Space Telescope

Polytrope: First Stellar Structure Model

Let's go back to hydrostatic equilibrium equation:

Rearrange $\frac{dP}{dr} = -G\frac{M(r)}{r^2}\rho(r) \quad \Rightarrow \frac{r^2}{\rho}\frac{dP}{dr} = -GM(r)$

Differentiate: $\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = -G\frac{dM}{dr} = -4\pi G\rho r^2$

Lane-Emden Equation results:

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = -\frac{G}{r^2}\frac{dM}{dr} = -4\pi G\rho$$

We have seen, under adiabatic/isentropic conditions that:

$$P \propto V^{-\gamma} \propto \rho^{-\gamma} \quad \text{Ideal Gas}$$

$P_e \propto n_e^{4/3,5/3} \propto \rho_e^{4/3,5/3}$ Deg. and Rel. Deg. electron gas

Motivated by these P-ρ relationships, the polytrope model adopts a pressure profile:

$$P = K\rho^{1+1/n}$$

And the density function is given by $\rho = \rho_c \phi^n$ where ϕ is a dimensionless function of radial coordinate r .

Put these into the Lane-Emden equation:

$$\frac{K(n+1)}{4\pi G \rho_c^{1-1/n}} \frac{1}{r^2} \frac{d}{dr}\left[r^2 \frac{d\phi}{dr}\right] = -\phi^n$$

Clean it up by setting $a^2 = \frac{K(n+1)}{4\pi G \rho_c^{1-1/n}}$ and $r = a\xi$

Finally, we have: $$\frac{1}{\xi^2} \frac{d}{d\xi}\left[\xi^2 \frac{d\phi}{d\xi}\right] = -\phi^n$$

Boundary conditions for function ϕ : $\phi(0) = 1$, $\left(\frac{d\phi}{d\xi}\right)_{\xi=0} = 0$

Mass: $$M = 4\pi \int_0^{R_*} \rho r^2 \, dr = 4\pi \rho_c a^3 \int_0^{\xi_*} \phi^n \xi^2 d\xi$$

$$= -4\pi \rho_c a^3 \int_0^{\xi_*} \frac{d}{d\xi} \left[\xi^2 \frac{d\phi}{d\xi}\right] d\xi$$

$$= -4\pi a^3 \rho_c \left(\xi^2 \frac{d\phi}{d\xi}\right)_{\xi_*}$$

$$= -4\pi \rho_c \left(\frac{K(n+1)}{4\pi G \rho_c^{1-1/n}}\right)^{3/2} \left(\xi^2 \frac{d\phi}{d\xi}\right)_{\xi_*}$$

Homework: Show that the case n=0, solution to the Lane-Emden equation is:

$$\phi = 1 - \xi^2/6$$

Polytrope Solutions

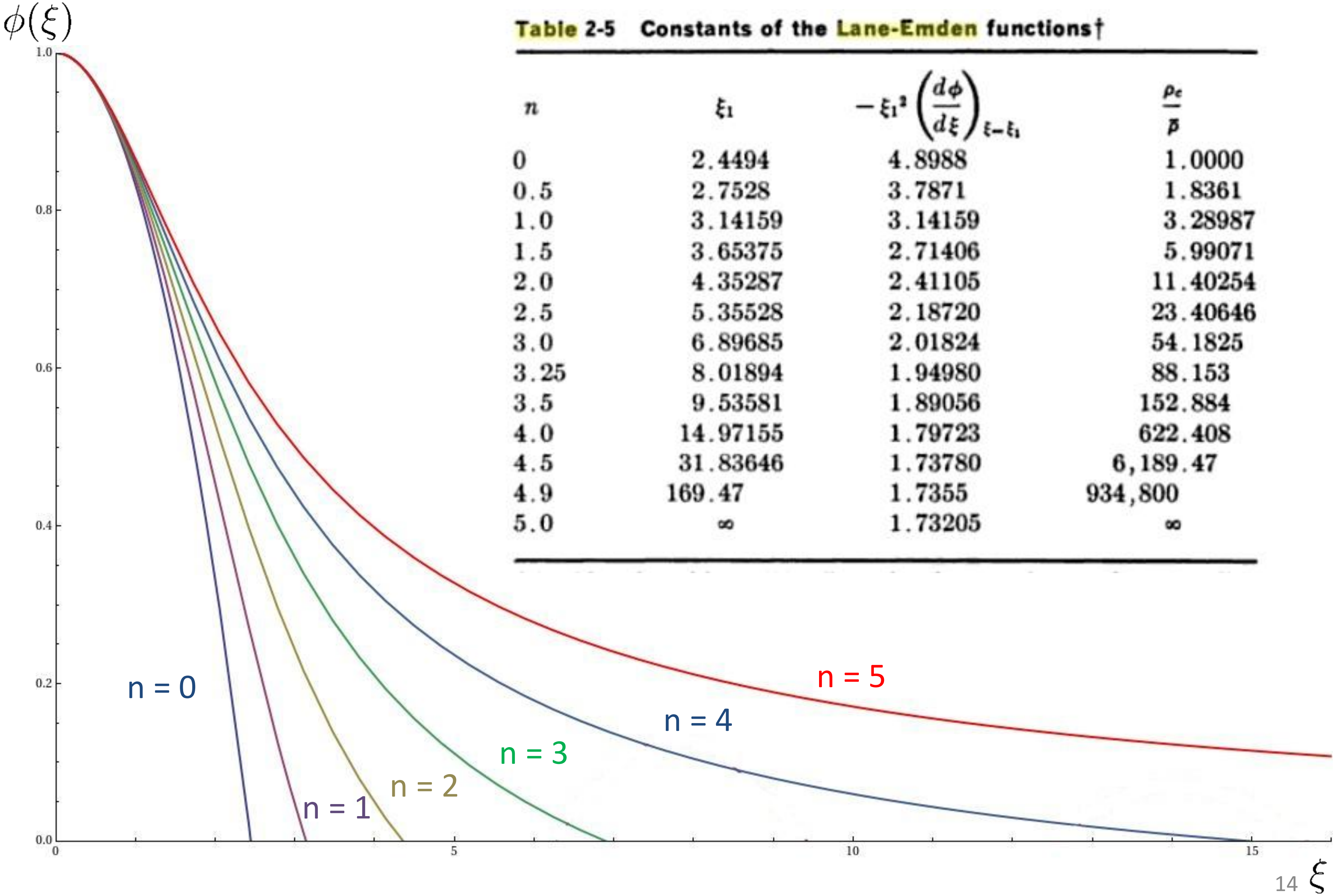


Table 2-5 Constants of the Lane-Emden functions†

n	ξ_1	$-\xi_1^2\left(\frac{d\phi}{d\xi}\right)_{\xi=\xi_1}$	$\frac{\rho_c}{\bar{\rho}}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6,189.47
4.9	169.47	1.7355	934,800
5.0	∞	1.73205	∞

Case n = 3: Relativistic Deg. Electron Gas and White Dwarfs

$$P_e^{rel} = \frac{\hbar c \pi^{2/3}}{12}(3n_e)^{4/3}$$

$$= \frac{\hbar c \pi^{2/3}}{12}\left(3\frac{\rho N_A}{\mu_e}\right)^{4/3} = K\rho^{1+1/3}$$

Therefore, we have:

$$K = \frac{\hbar c \pi^{2/3}}{12}\left(3\frac{N_A}{\mu_e}\right)^{4/3}$$

$$M = -4\pi\rho_c \left(\frac{K(n+1)}{4\pi G\rho_c^{1-1/n}}\right)^{3/2} \left(\xi^2 \frac{d\phi}{d\xi}\right)_{\xi_*}$$

When n = 3 $$M = -4\pi \left(\frac{K}{\pi G}\right)^{3/2} \left(\xi^2 \frac{d\phi}{d\xi}\right)_{\xi_*}$$

Substitute in for K from previous page:

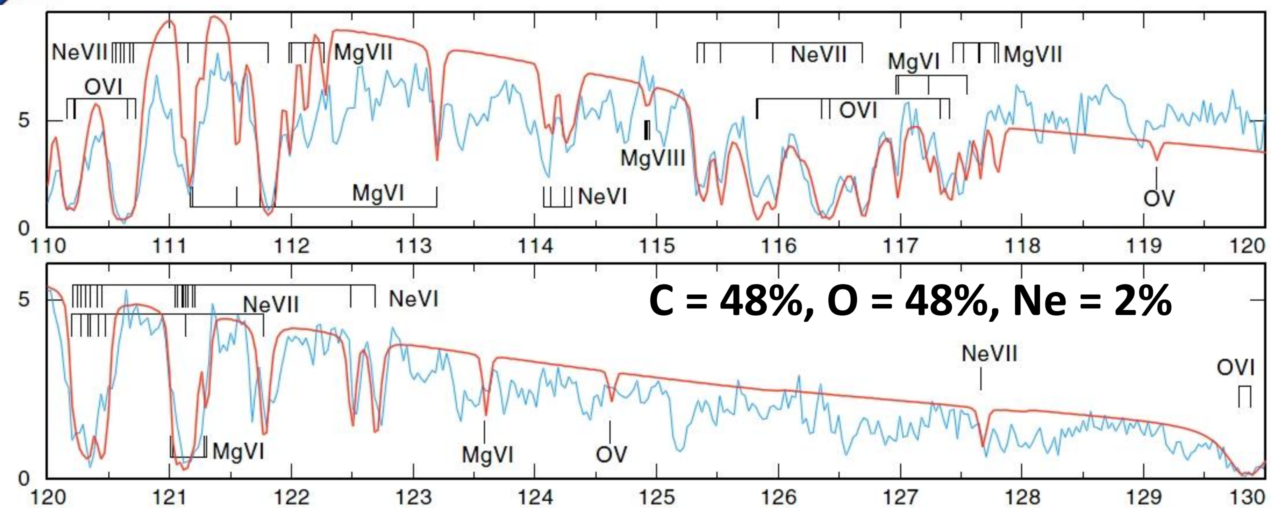
$$M = -36\pi^2 \left(\frac{\hbar c}{12\pi G}\right)^{3/2} \left(\frac{N_A}{\mu_e}\right)^2 \left(\xi^2 \frac{d\phi}{d\xi}\right)_{\xi_*}$$

$$= \frac{5.81}{\mu_e^2} M_\odot$$

Where μ_e is in grams.

This is Chandrasekhar's mass relation for White Dwarf's, and now you've seen how it is derived! ☺

Chandra X-ray Spectrum of WD H1504+65



What is a mass estimate of this Oxygen-Neon WD?

$$\frac{1}{\mu_e} = \sum_{z=6}^{10} \frac{X_z}{A_z}(n_z - 1) = \sum_{z=6}^{10} \frac{X_z}{A_z} z \qquad n_z - 1 = z \text{ full ionization}$$

$$= 0.48 \times 6/12 + 0.48 \times 8/16 + 0.02 \times 10/20 = 0.49$$

$$M_{WD} = \frac{5.81}{\mu_e^2} M_\odot = 5.81 \times 0.49^2 M_\odot = 1.39 M_\odot$$

Exercise for student: Show that the ratio of mean to central density is (for any index n):

$$\frac{\bar{\rho}}{\rho_c} = -3\left(\frac{1}{\xi}\frac{d\phi}{d\xi}\right)_{\xi_*}$$

Central Pressure: Use equations on page 29 for a and $R_* = a\xi_*$, along with $P_c = K\rho_c^{1+1/n}$ and the result for Mass on page 30.

Exercise for student: show that

$$P_c = \frac{1}{4\pi(n+1)(d\phi/d\xi)^2_{\xi_*}}\frac{GM_*^2}{R_*^4}$$

For n = 3: $$P_c = 1.24 \times 10^{11}\left(\frac{M_*}{M_\odot}\right)^2\left(\frac{R_\odot}{R_*}\right)^4 \text{ atm}$$

Case n = 3: "Main Sequence"

Main sequence stars have both particle pressure and photon pressure acting within their interiors.

$$P_{gas} = n\tau \qquad\qquad P_\gamma = \frac{\pi^2}{45\hbar^3 c^3}\tau^4$$

Total Pressure: $P_{tot} = P_{gas} + P_\gamma$

Suppose gas pressure contributes a fraction β to the total pressure . Then the photon pressure is $P_\gamma = (1-\beta)P_{tot}$. And $P_{gas} = \beta P_{tot}$

In thermodynamic equilibrium, both gases must have the same temperature τ . Eliminating the common temperature, we have (exercise for student)

$$\chi = \frac{\pi^2}{45\hbar^3 c^3} \qquad P = \left(\frac{1-\beta}{\chi\beta^4}\right)^{1/3}\left(\frac{N_A}{\mu}\right)^{4/3}\rho^{4/3}$$

Polytrope index of 3 again!

From $P = K\rho^{1+1/n} \rightarrow K = \left(\frac{1-\beta}{\chi\beta^4}\right)^{1/3}\left(\frac{N_A}{\mu}\right)^{4/3}$

This value of K can be used in mass expression (page 13) to determine Main sequence masses. For **you** to show that the Mass is given by:

$$M = -\frac{4\pi}{(\pi G)^{3/2}\chi^{1/2}} \frac{\sqrt{1-\beta}}{\beta^2} \left(\frac{N_A}{\mu}\right)^2 \left(\xi^2 \frac{d\phi}{d\xi}\right)_{\xi_*}$$

$$= 18.0 \frac{\sqrt{1-\beta}}{\mu^2 \beta^2} M_\odot$$

Finally, let's get the central temperature of this Main Sequence model:

$$P_{gas} = \frac{N_A}{\mu} \rho T = \beta P_{tot}$$

$$\Rightarrow \; T_c = \beta \frac{\mu}{N_A k} \frac{P_c}{\rho_c}$$

(From previous page P/ρ)

$$= \beta \frac{\mu}{N_A k} \left(\frac{1-\beta}{\chi \beta^4}\right)^{1/3} \left(\frac{N_A}{\mu}\right)^{4/3} \rho_c^{1/3}$$

$$= 4.6 \times 10^6 \mu \beta \left(\frac{M_*}{M_\odot}\right)^{2/3} \rho_c^{1/3}$$

In terms of the mean density, using $\frac{\bar{\rho}}{\rho_c} = -3 \left(\frac{1}{\xi} \frac{d\phi}{d\xi} \right)_{\xi_*}$ from page 35 and table on page 31:

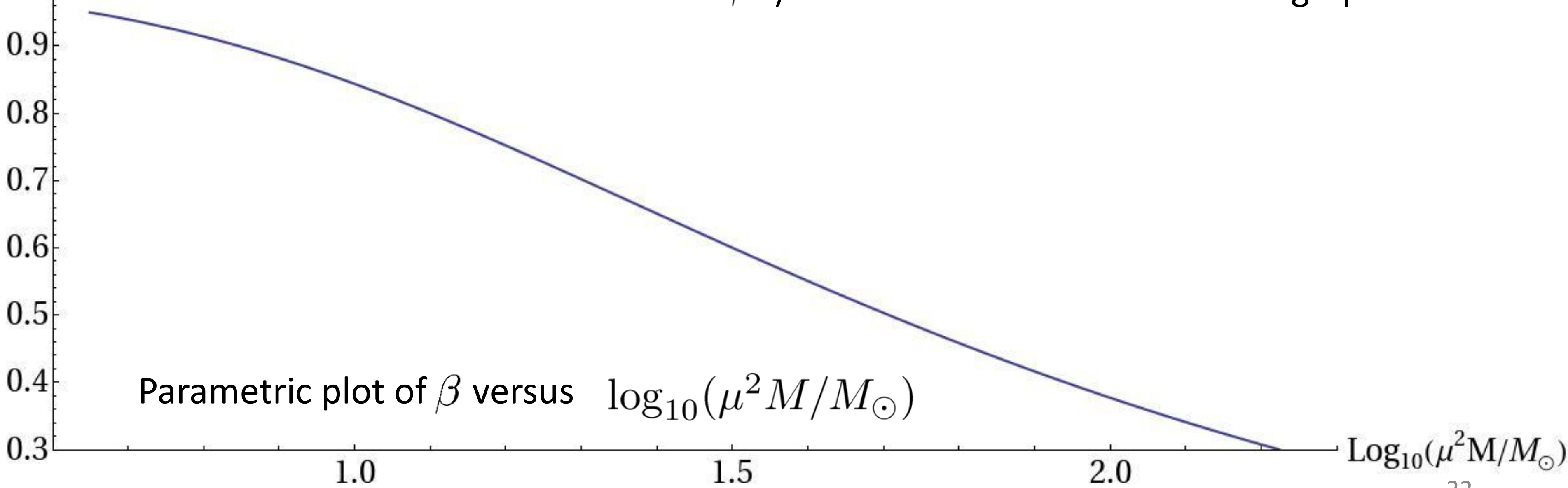
$$T_c = 17.4 \times 10^6 \mu\beta \left(\frac{M_*}{M_\odot} \right)^{2/3} \bar{\rho}^{1/3}$$

Homework: A certain star has the following properties:

$$M = 30 M_{\odot} \qquad R = 6.6 R_{\odot} \qquad X = 0.70 \qquad Y = 0.30$$

Using these values, graph below, and the previous equation, estimate the central temperature using the previous result for the polytrope Standard Model.

Refer to page 37 for the final mass formula of the Main Sequence Star model to understand why the plot is parameterized. (We do **not** Isolate β in the mass formula. We isolate $\mu^2 M/M_{\odot}$ and plot its values for values of β) And this is what we see in the graph.



Parametric plot of β versus $\log_{10}(\mu^2 M/M_{\odot})$