Nuclear Astrophysics

Lecture 2

Thurs. Oct. 27, 2011

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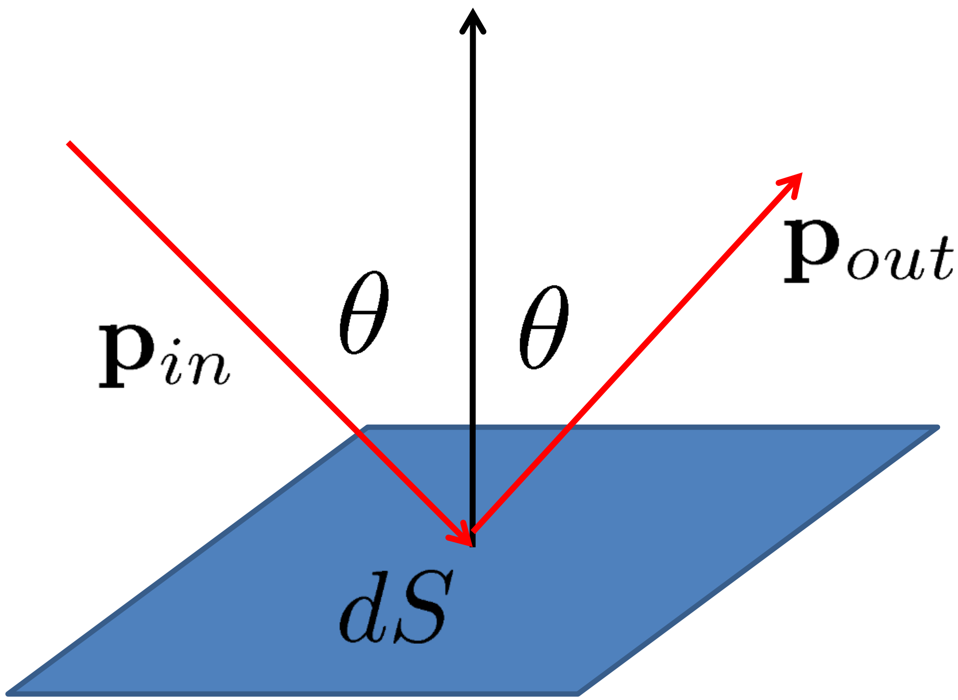
THERMODYNAMIC PROPERTIES: EQUATIONS OF STATE

On the Road to Pressure Integral

Momentum transfer imparted to surface dS:

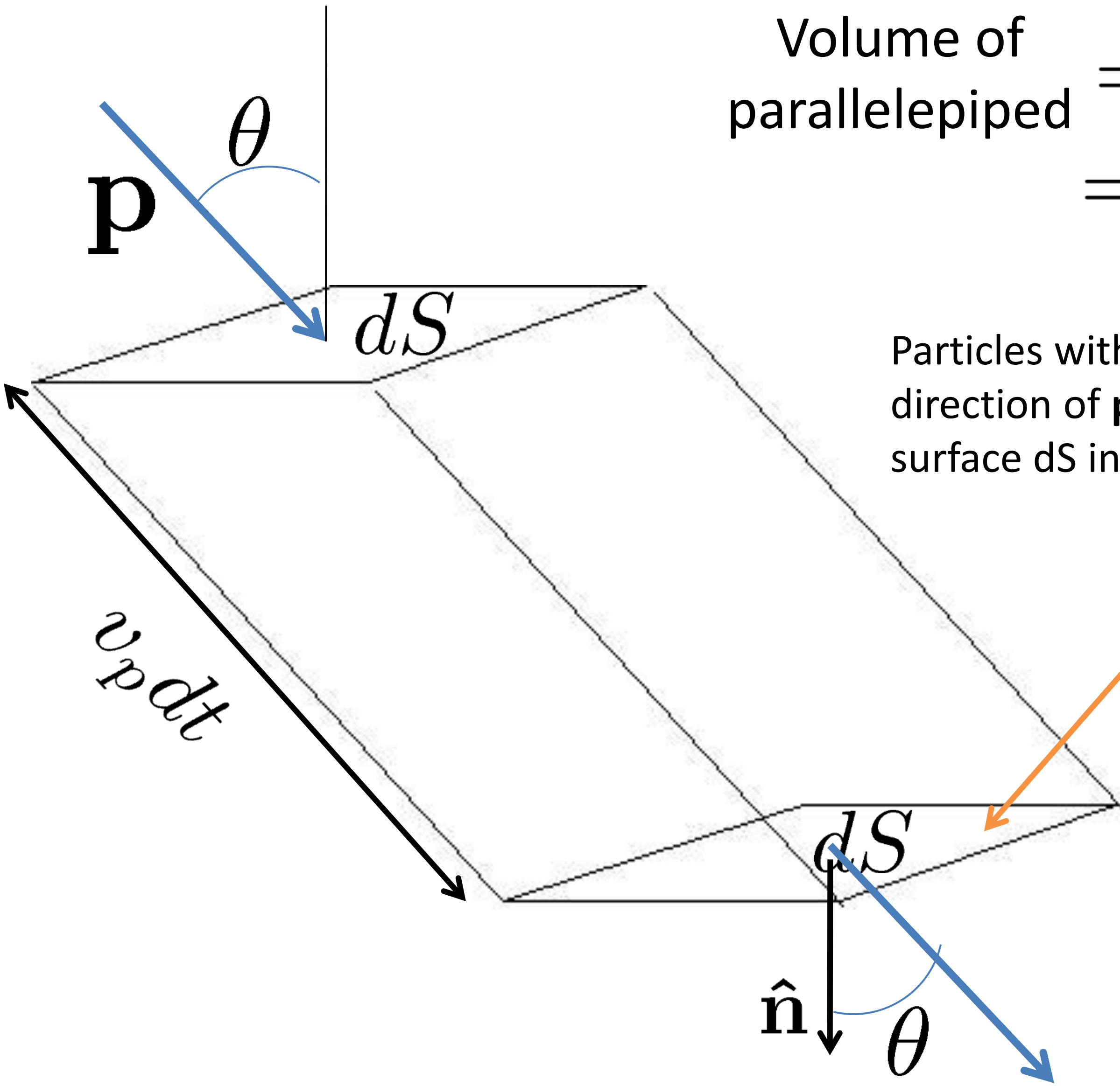
$$\Delta \mathbf{p} = 2p\cos\theta\,\hat{\mathbf{z}}$$



Flux Volume

Volume of parallelepiped $= \mathbf{v}_p dt \cdot \hat{\mathbf{n}} dS$

$= v_p \cos\theta \, dt \, dS$



Particles within this volume, moving in direction of **p**, can strike the bottom surface dS in a unit time

Pressure Integral

Number of all particles with momentum between p and p + dp, θ and θ + dθ , ϕ and ϕ + dϕ striking dS will be:

Number per unit vol with momentum p X Flux Volume X Fraction of solid angle subtended by dS

Total hitting dS $= N_{hit} = n(p) \times v_p \cos\theta \, dt \, dS \times \frac{\Delta\Omega}{4\pi}$

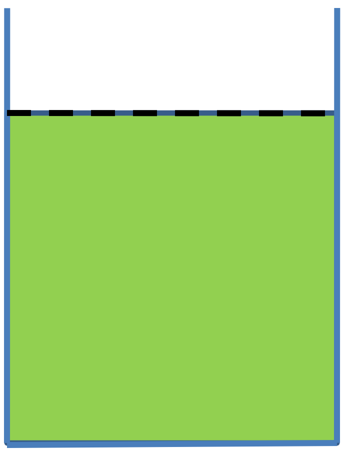
$$P = N_{hit} \frac{1}{dS} \frac{d\mathbf{p}}{dt} = N_{hit} \frac{1}{dS} \frac{2p\cos\theta}{dt}$$

$$P = \int_0^\infty p v_p n(p) dp \int_0^{\pi/2} \cos^2\theta \sin\theta d\theta$$

$$= \frac{1}{3} \int_0^\infty p v_p n(p) dp$$

Electron in a box: Length L

Schroedinger Wave Equation: $-\frac{\hbar^2}{2m}\nabla^2\Psi = \epsilon_n\Psi$



ϵ_F

$$\epsilon_n = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2(n_x^2+n_y^2+n_z^2)$$
$$= \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n^2$$

Fill the box with N electrons. Each ϵ_n level can take 2 electrons (Pauli). Electrons fill box from lowest energy level up to a maximum energy level we call ϵ_F

Only positive integers $n_{x,y,z}$ are valid. States in the positive octant (1/8) of sphere in n_x, n_y, n_z coordinates contribute. Volume of this sphere: $4\pi n_F^3/3$, n_F is the radius of sphere at the Fermi Level with energy ϵ_F . N can now be related to the number of levels

$$N = 2\times\frac{1}{8}\times\frac{4\pi}{3}n_F^3 = \frac{\pi}{3}n_F^3$$

Energy of Electron Gas in Box

$$U = 2 \sum_{n \le n_F} \epsilon_n = 2 \times \frac{1}{8} \times 4\pi \int_0^{n_F} \epsilon_n n^2 dn \qquad \epsilon_n = \frac{\hbar^2}{2m} \left(\frac{\pi}{L}\right)^2 n^2$$

$$= \frac{\pi^3}{10m} \left(\frac{\hbar}{L}\right)^2 n_F^5 = \frac{\pi^3}{10m} \left(\frac{\hbar}{L}\right)^2 \left(\frac{3N}{\pi}\right)^{5/3} \qquad N = \frac{\pi}{3} n_F^3$$

The pressure of this electron gas is then given by (and $V = L^{1/3}$)

$$P_e = -\frac{\partial U}{\partial V} = \frac{\pi^3}{15m} \hbar^2 \left(\frac{3n_e}{\pi}\right)^{5/3}$$

Where $n_e = N/V$ is the electron number concentration.

Pressure **NOT** dependent on temperature, but only on the electron volume concentration! Degenerate electron gas does **NOT** behave like an ideal gas.

Relativistic Degenerate Electron Gas

We previously had on page 7 for kinetic energy of electron in orbital "n":

$$\epsilon_n = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n^2$$

Let us equate this to $p^2/2m$ and get: $p = \hbar\pi n/L = \hbar\pi n/V^{1/3}$

For the pressure integral, we need $n(p) \equiv \frac{dN}{V\,dp}$

We also had, from page 7, for the total number of electrons in the box: $N = \frac{\pi}{3} n_F^3$

These two equations give us p as a function of N, by substituting for n:

$$p = \frac{\hbar\pi}{V^{1/3}}\left(\frac{3N}{\pi}\right)^{1/3}$$

Solve for N, and take dN/dp. Result is: $\frac{dN}{V\,dp} = \frac{p^2}{\pi^2\hbar^3} \equiv n(p)$

For relativistic electrons, $v \approx c$ and the pressure integral becomes:

$$P_e^{rel} = \frac{1}{3}\int_0^{p_F} pc\frac{p^2}{\pi^2\hbar^3}dp$$
$$= \frac{c}{12\pi^2\hbar^3}p_F^4$$

From previous page, we have all we need to get the Fermi momentum, p_F

$$p_F = \hbar\pi^{2/3}(3n_e)^{1/3}$$

Finally, we have the relativistic degenerate electron pressure:

$$P_e^{rel} = \frac{\hbar c\pi^{2/3}}{12}(3n_e)^{4/3}$$

THE PATH TO RADIATION PRESSURE

Photon Gas in Box: Length L

Maxell's Equation for the Electric field: $c^2\nabla^2\mathbf{E} = \frac{\partial^2}{\partial t^2}\mathbf{E}$

$$E_x = E_{x_0} \sin\omega t \;\; \cos\frac{n_x\pi x}{L} \;\; \sin\frac{n_y\pi y}{L} \;\; \sin\frac{n_z\pi z}{L}$$

With E_y and E_z given by similar expressions by cyclic shifting of the cosine in the right (→) direction

Substitution of these back into Maxell's Eqn yields:

$$c^2\pi^2(n_x^2 + n_y^2 + n_z^2) = c^2\pi^2 n^2 = \omega^2 L^2 \qquad \rightarrow \omega_n = \frac{n\pi c}{L}$$

Mode Energy: $\epsilon_n = \hbar\omega_n = \frac{n\hbar\pi c}{L} = \frac{n\hbar\pi c}{V^{1/3}}$

Each mode will have some average number of photons in it. Call this number .
We need to eventually find a formula for S

Total energy of the photon gas: $U = \sum_{n=1}^{\infty} s_n \epsilon_n = \sum_{n=1}^{\infty} \hbar s_n \omega_n$

Photon pressure: $$P_\gamma = \frac{\partial U}{\partial V} = \sum_{n=1}^{\infty} \hbar s_n \frac{\partial \omega_n}{\partial V} = \frac{1}{3V} \sum_{n=1}^{\infty} \hbar s_n \omega_n$$

$$= \frac{U}{3V}$$

Thermodynamic Partition Function Z for a mode (review your thermodynamics)

$$Z = \sum_{s=0}^{\infty} \exp(-s\hbar\omega/\tau) = \sum_{s=0}^{\infty} x^s, \; x \equiv \exp(-\hbar\omega/\tau)$$

$$= \frac{1}{1-x} = \frac{1}{1-\exp(-\hbar\omega/\tau)}$$

Energy in a mode: $\langle \epsilon_n \rangle = \tau^2 \frac{\partial}{\partial \tau} \ln Z$

$$= \frac{\hbar\omega}{\exp(\hbar\omega/\tau)-1}$$

Total Energy: $$U = \sum_n \langle \epsilon_n \rangle = \sum_n \frac{\hbar\omega_n}{\exp(\hbar\omega_n/\tau) - 1}$$

Remember from page 12, that n is a triplet of integers: n_x, n_y, n_z. We replace the sum by an integral over dn_x, dn_y, dn_z, and change to spherical coordinates:

$$U = 2 \times \frac{1}{8} \int_0^\infty 4\pi n^2 dn \hbar \frac{\omega_n}{\exp(\hbar\omega_n/\tau) - 1}$$

2 polarizations

And we recall from page 12 that $\omega_n = n\pi c/V^{1/3}$

$$U = \pi^2 \hbar c/V^{1/3} \int_0^\infty n^3 dn \frac{1}{\exp(\hbar\omega_n/\tau) - 1}$$

$$= (\pi^2 \hbar c/L)(\tau L/\pi\hbar c)^4 \int_0^\infty dx \frac{x^3}{\exp x - 1}$$

Change variable $x = \pi\hbar c n/L\tau$

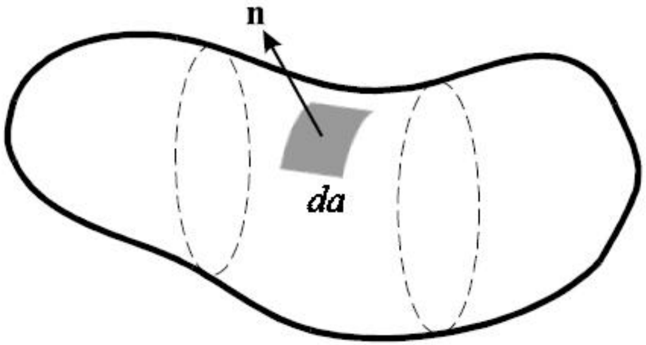
$$= \frac{\pi^2 V}{15\hbar^3 c^3} \tau^4$$

(the integral $= \pi^4/15$)

Pressure (at last!) was found, on page 13, to be U/3V: $$P_\gamma = \frac{\pi^2}{45\hbar^3 c^3}\tau^4$$

Hydrostatic Equilibrium

The gravitational body forces integrated over the volume V must be balanced by the pressure acting on the total surface area A.



$$\int_V \mathbf{g}\,\rho\, dV = \int_A P\, d\mathbf{a} \qquad (*)$$

Gauss' Law (Divergence Theorem): $\int_V (\nabla \cdot \mathbf{F}) dV = \int_A \mathbf{F} \cdot d\mathbf{a}$

Use $\mathbf{F} = f\mathbf{v}$, where **v** is a constant vector (has no (x,y,z) dependence) and also use the following identity: $\nabla \cdot (f\mathbf{v}) = f(\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla f$

$= 0$

Now Div. Thm: $\int_V (\nabla \cdot f\mathbf{v}) dV = \mathbf{v} \cdot \int_V \nabla f\, dV = \mathbf{v} \cdot \int_A f\, d\mathbf{a}$

$$\Rightarrow \mathbf{v} \cdot \left(\int_V \nabla f\, dV - \int_A f\, d\mathbf{a} \right) = 0$$

But **v** was chosen to be a constant (non-zero) field. So, term in ( ) is zero. Use it on RHS in force balance equation (*).

Force balance: $$\int_V \mathbf{g}\,\rho\,dV = \int_A P\,d\mathbf{a}$$

$$= \int_V \nabla P\,dV$$ (From last identity on previous page)

$$\Rightarrow \int_V \mathbf{g}\,\rho\,dV - \int_V \nabla P\,dV = 0$$

This result must hold throughout the entire volume V, and since volume shape is arbitrary, we therefore have:

$$\Rightarrow \mathbf{g}\rho = \nabla P$$

In spherical geometry, using $g = -G\dfrac{M}{r^2}$, where M is the <u>enclosed</u> mass inside radius r, we finally have:

$$\boxed{\frac{dP}{dr} = -G\frac{M(r)}{r^2}\rho(r)}$$

Summary of Results so far.

$$\frac{dP}{dr} = -G\frac{M(r)}{r^2}\rho(r)$$

$$P_{gas} = n\tau \qquad , n = N/V$$

$$P_e = -\frac{\partial U}{\partial V} = \frac{\pi^3}{15m}\hbar^2\left(\frac{3n_e}{\pi}\right)^{5/3}$$

$$n_e = N_e/V$$

$$P_e^{rel} = \frac{\hbar c \pi^{2/3}}{12}(3n_e)^{4/3}$$

$$P_\gamma = \frac{\pi^2}{45\hbar^3c^3}\tau^4$$