Nuclear Astrophysics

Lecture 10

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Summary of Reaction Rate

$$\boxed{r_{12} = \frac{2.62 \times 10^{29} \rho^2}{1 + \delta_{12}} \frac{X_1 X_2}{A_1 A_2} \frac{S_0}{\mu Z_1 Z_2} \chi^2 e^{-\chi}}$$

$$\boxed{r_{12}(T) = r_{12}(T_0) \left(\frac{T}{T_0} \right)^{(-2+3\frac{E_{\text{eff}}}{\tau})/3}}$$

$$\chi = 3E_{\text{eff}}/\tau \qquad E_{\text{eff}} = \left(\frac{b\tau}{2} \right)^{2/3} = 1.22 (Z_1^2 Z_2^2 \mu T_6^2)^{1/3} \text{ keV}$$

$$\tau = kT = 0.086 T_6$$

Energy Generation & Standard Model

Let us consider a representation for the nuclear rate of energy production (in units of energy per unit time, per unit mass) to be given by the following mathematical form:

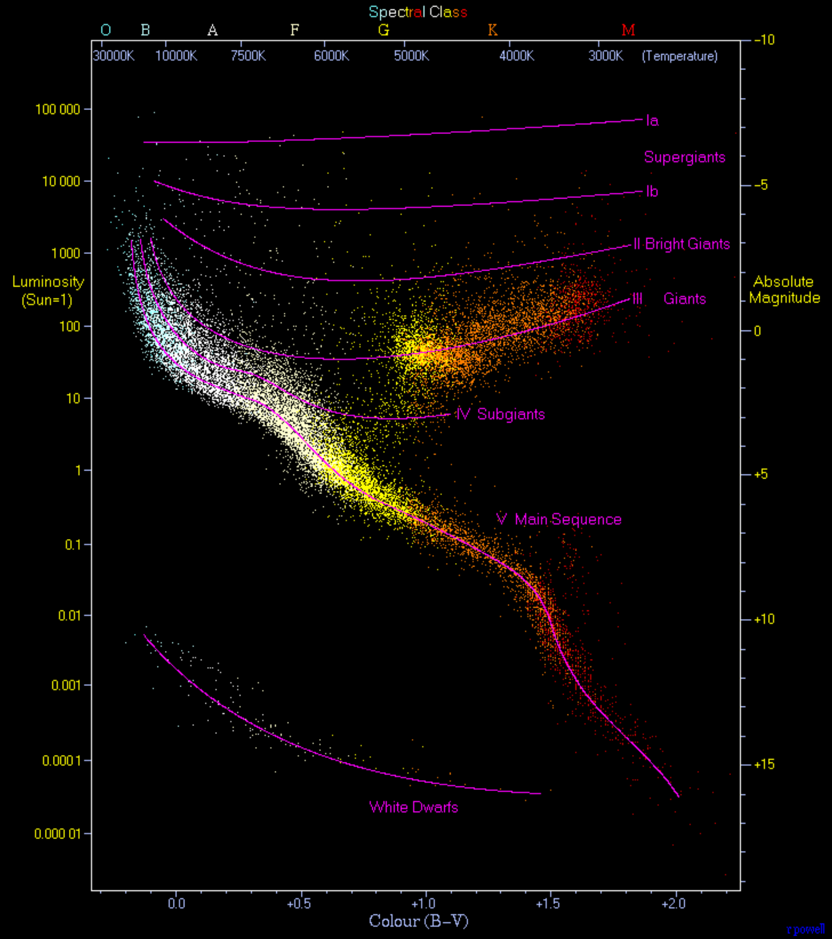
$$\epsilon \propto \rho^{u-1} T^{s}$$

Then we can, quite generally, also write:

$$\frac{\epsilon}{\epsilon_c} = \left(\frac{\rho}{\rho_c}\right)^{u-1} \left(\frac{T}{T_c}\right)^{s}$$

For a Polytrope model, once the solution ϕ to Lane-Emden equation is known, then we have the following results (for n = 3 Polytrope):

$$\boxed{\frac{\rho(r)}{\rho_c} = \phi^3(r) \qquad \frac{T(r)}{T_c} = \phi(r) \qquad \frac{P(r)}{P_c} = \phi^4(r)}$$



HERTZSPRUNG-RUSSEL DIAGRAM & NUCLEAR BURNING

Proton-Proton-Chain

$$p + p \rightarrow d + e^+ + \nu$$

$$p + d \rightarrow {}^3He + \gamma$$

86% | 14%

$${}^3He + {}^3He \rightarrow {}^4He + 2p$$

PP-I

Q_{eff}= 26.20 MeV

$${}^3He + {}^4He \rightarrow {}^7Be + \gamma$$

99.7% | 0.3%

$${}^7Be + e^- \rightarrow {}^7Li + \nu$$

$${}^7Li + p \rightarrow 2\ {}^4He$$

PP-II

Q_{eff}= 25.66 MeV

$${}^7Be + p \rightarrow {}^8B + \gamma$$

$${}^8B \rightarrow {}^8Be + e^+ + \nu$$

$$\downarrow$$

$$2\ {}^4He$$

PP-III

Q_{eff}= 19.17 MeV

Netto: $4p \rightarrow {}^4He + 2e^+ + 2\nu + Q_{eff}$

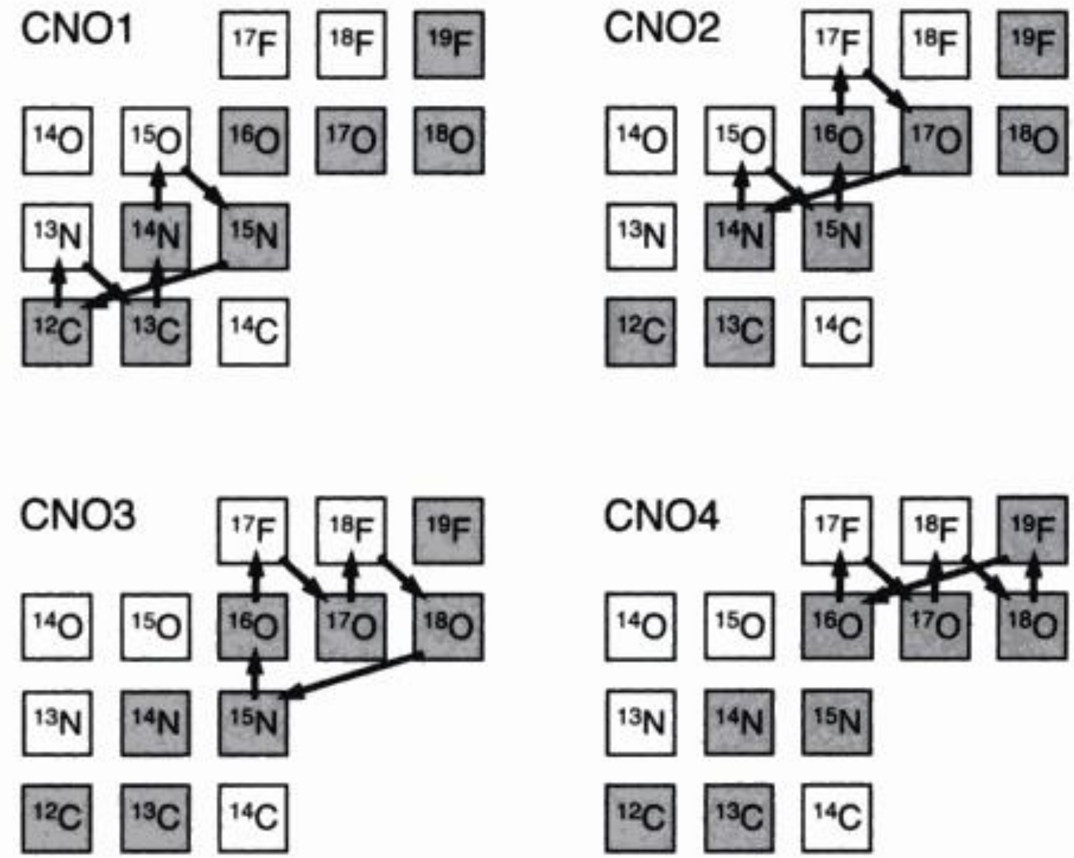
CNO Nuclear Cycles

CNO burning cycles for hydrogen under non-explosive conditions.

Each cycle converts 4 protons into an alpha particle.

It is these cycles that nuclear burning enters once proton-proton and ^{4}He burning (by the triple-alpha process) have built up enough ^{12}C abundance.

Remember: These form cycles because, at some point along the path, we encounter a nucleus “Y” that has a binding energy for alpha particles that is ***lower*** than the minimum excitation energy of “Y” when it is made by proton capture. Example in CNO3:
$^{18}O(p,\gamma)^{19}F^{*} \rightarrow \alpha + {}^{15}N$



CNO1	CNO2	CNO3	CNO4
$^{12}C(p,\gamma)^{13}N$	$^{14}N(p,\gamma)^{15}O$	$^{15}N(p,\gamma)^{16}O$	$^{16}O(p,\gamma)^{17}F$
$^{13}N(\beta^{+}\nu)^{13}C$	$^{15}O(\beta^{+}\nu)^{15}N$	$^{16}O(p,\gamma)^{17}F$	$^{17}F(\beta^{+}\nu)^{17}O$
$^{13}C(p,\gamma)^{14}N$	$^{15}N(p,\gamma)^{16}O$	$^{17}F(\beta^{+}\nu)^{17}O$	$^{17}O(p,\gamma)^{18}F$
$^{14}N(p,\gamma)^{15}O$	$^{16}O(p,\gamma)^{17}F$	$^{17}O(p,\gamma)^{18}F$	$^{18}F(\beta^{+}\nu)^{18}O$
$^{15}O(\beta^{+}\nu)^{15}N$	$^{17}F(\beta^{+}\nu)^{17}O$	$^{18}F(\beta^{+}\nu)^{18}O$	$^{18}O(p,\gamma)^{19}F$
$^{15}N(p,\alpha)^{12}C$	$^{17}O(p,\alpha)^{14}N$	$^{18}O(p,\alpha)^{15}N$	$^{19}F(p,\alpha)^{16}O$

$T_{1/2}$: ^{13}N (9.965 min); ^{15}O (122.24 s); ^{17}F (64.49 s); ^{18}F (109.77 min)

The nuclear energy production in Main Sequence stars comes either from the proton-proton chain **or** from the CNO cycle. (C = Carbon, N = Nitrogen, O = Oxygen).

PP-Chain Parameters

$$\epsilon_{pp} = \rho X^2 \epsilon_{0_{pp}} \left(\frac{T}{T_0}\right)^{n_{pp}}$$

$$n_{pp} = 4.6$$

$$X = 0.71$$

$$\epsilon_{0_{pp}} = 0.068 \text{ erg g}^{-1} \text{ s}^{-1}$$

$$T_0 = 10^7$$

CNO-Cycle Parameters

$$\epsilon_{CNO} = \rho X X_{CNO} \epsilon_{0_{CNO}} \left(\frac{T_6}{25}\right)^{n_{CNO}}$$

$$n_{CNO} = 16.7$$

$$X_{CNO} = \frac{2}{3} Z \qquad Z = 0.02$$

$$\epsilon_{0_{CNO}} = 2.2 \times 10^4 \text{ erg s}^{-1}\text{g}^{-1}$$

From Lecture 2, we derived an expression for the central temperature of a Polytrope star as,

$$T_c = 4.6 \times 10^6 \mu\beta \left(\frac{M_*}{M_\odot}\right)^{2/3} \rho_c^{1/3}$$

Central density: $\Rightarrow \rho_c = \dfrac{T_c^3}{(4.6 \times 10^6 \ \mu\beta)^3 \ m^2}$ $\qquad m \equiv \dfrac{M_*}{M_\odot}$

$$= 10^{21} \frac{T_{7c}^3}{(4.6 \times 10^6 \ \mu\beta)^3 \ m^2} \qquad T_{7c} = \frac{T_c}{10^7}$$

In these formulae, the temperature is in kelvin, and density is grams/cm^3.

Numerically, we find that: $\boxed{\rho_c = F \dfrac{T_{7c}^3}{(\mu\beta)^3 \ m^2}}$ $\qquad F = 10.27$

PP-Chain Nuclear Energy Rate vs Core Temperature

From page 17, we had: $\epsilon_{pp} = \rho X^2 \epsilon_{0_{pp}} \left(\frac{T}{T_0}\right)^{n_{pp}}$

Dropping the "pp" subscripts, and considering this formula at the centre of the star:

$$\epsilon_c = \rho_c X^2 \epsilon_0 \left(\frac{T_c}{T_0}\right)^n = \rho_c X^2 \epsilon_0 T_{7c}^n$$

(Remembering $T_0 = 10^7$)

Substitute in the previous result for ρ_c :

$$\boxed{\Rightarrow \epsilon_c = F \frac{\epsilon_0 X^2}{(\mu\beta)^3 \; m^2} T_{7c}^{n+3}}$$

CNO-Cycle Luminosity vs Core Temperature

From page 17, we had: $\epsilon_{CNO} = \rho X X_{CNO} \epsilon_{0_{CNO}} \left(\frac{T_6}{25} \right)^{n_{CNO}}$

Dropping the "CNO" subscripts, and considering this formula at the centre of the star:

$$\epsilon_c = \rho X X_{CNO} \epsilon_0 \left(\frac{10}{25} \right)^n T_{7c}^n$$

Substitute in the previous result for ρ_c : $\boxed{\epsilon_c = F \frac{\rho X X_{CNO} \epsilon_0}{(\mu\beta)^3 m^2} \left(\frac{10}{25} \right)^n T_{7c}^{n+3}}$

Last lecture, we derived, for Polytrope (n = 3) the result: $L = M_* \frac{\langle \epsilon \rangle}{\epsilon_c} \epsilon_c$

$$\frac{\langle \epsilon \rangle}{\epsilon_c} = \frac{3.23}{(3u+n)^{3/2}}$$

Refer to Page 2 to understand that $u = 2$.

The power per unit mass produced in the stellar core is, after all, what is responsible for the luminosity at the stellar surface. We now try to come up with a generalized formula for the nuclear energy generation rate for any Main Sequence star.

We do this by performing a mass-averaged energy generation rate. And this mass-averaged rate times the stellar mass should, if the model is anywhere close to reality, give us a quantitative trend that is similar to the observational data.

$$L_* = M_* \langle \epsilon \rangle = M_* \left(\frac{\langle \epsilon \rangle}{\epsilon_c} \right) \epsilon_c$$

We now have all the ingredients we need to determine the relationship between surface temperature (effective temperature) and the stellar luminosity. Start substituting the results from previous slides. I will do the case for the pp-chain

$$L_* = M_* \frac{3.23}{(3u+n)^{3/2}} \epsilon_c$$

$$= F \frac{3.23}{(3u+n)^{3/2}} \frac{M_* \epsilon_0 X^2}{(\mu\beta)^2 \; m^2} T_{7c}^{n+3}$$

Multiply and divide by unity: $M_\odot / M_\odot$ so that numerator will have $M_* \rightarrow m$

Continuing: $$L_* = F M_\odot \frac{3.23}{(3u+n)^{3/2}} \frac{\epsilon_0 X^2}{(\mu\beta)^3\ m} T_{7c}^{n+3}$$

Divide both sides by Solar Luminosity, $L_\odot$, and use $M_\odot / L_\odot = 0.52$ and $F = 10.27$ and $\epsilon_0 = 0.068$. Then, we have:

$$\boxed{\frac{L_*^{pp}}{L_\odot} = 1.18 \frac{X^2}{m(\mu\beta)^3\ (3u+n)^{3/2}} T_{7c}^{n+3}}$$

pp-chain burning

$n = n_{pp} = 4.6$

This result is the relationship between the luminosity of the star in terms of its **core** temperature, at the centre of the star, for **proton-proton** nuclear burning.

Similar steps will lead you to a corresponding formula for luminosity in terms of core temperature for stars with the nuclear energy generation rate dominated by **CNO-Cycle** burning. (And you should do this: you have all the formula from page 27 and 28 to do it).

$$\boxed{\frac{L_*^{CNO}}{L_\odot} = 3.79 \times 10^5 \frac{X\ X_{CNO}}{m(\mu\beta)^3\ (3u+n)^{3/2}} \left(\frac{10}{25}\right)^n T_{7c}^{n+3}}$$

$n = n_{CNO} = 16.7$

From page 15, Lecture 6, the Main Sequence Mass-Luminosity relationship was derived. We found that:

$$\frac{L_*}{L_\odot} = 103 \frac{(\mu\beta)^7\, m^5}{m^2(\mu\beta)^3 + 5.94} \ \text{erg/s} \qquad (*)$$

And we have just found that the nuclear burning for pp-chains and CNO cycle give:

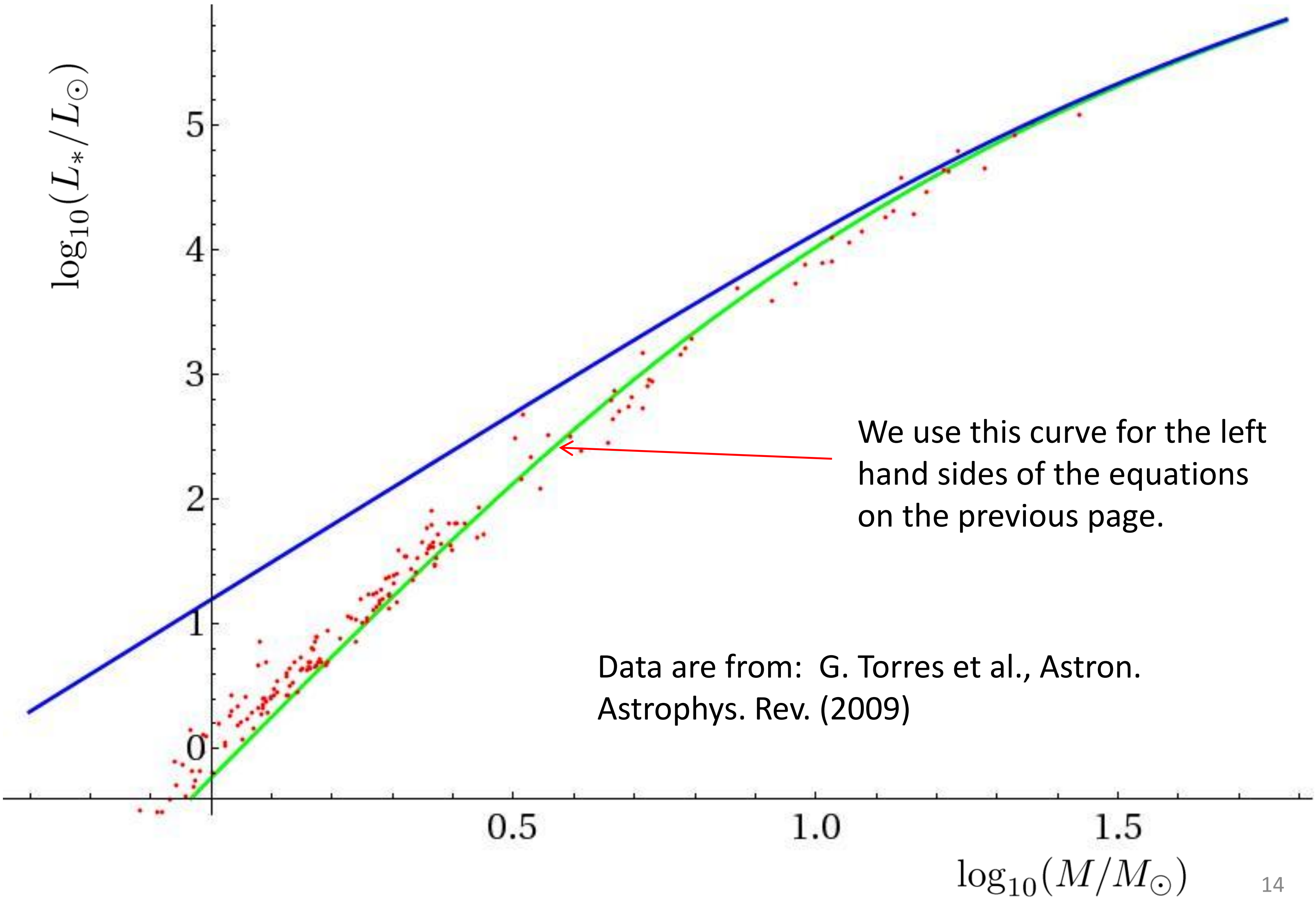
$$\frac{L_*^{pp}}{L_\odot} = 1.18 \frac{X^2}{m(\mu\beta)^3\ (3u+n)^{3/2}} T_{7c}^{n+3} \qquad n = n_{pp} = 4.6 \qquad (**)$$

$$\frac{L_*^{CNO}}{L_\odot} = 3.79 \times 10^5 \frac{X\ X_{CNO}}{m(\mu\beta)^3\ (3u+n)^{3/2}} \left(\frac{10}{25}\right)^n T_{7c}^{n+3} \qquad n = n_{CNO} = 16.7 \qquad (***)$$

Solar abundances give, (refer also to page 5): $X = 0.71,\ \ Z = 0.02$, $X_{CNO} = 2Z/3$ $\mu = 0.61$. Note: The Eddington Quartic Equation (page 37, Lect. 2/3) lets us solve for β for a **chosen** stellar mass m.

Once we have β and m we can equate (**) and (***) with (*) and get the stellar core temperature T_{7c} of Main Sequence stars in terms of the stellar mass.

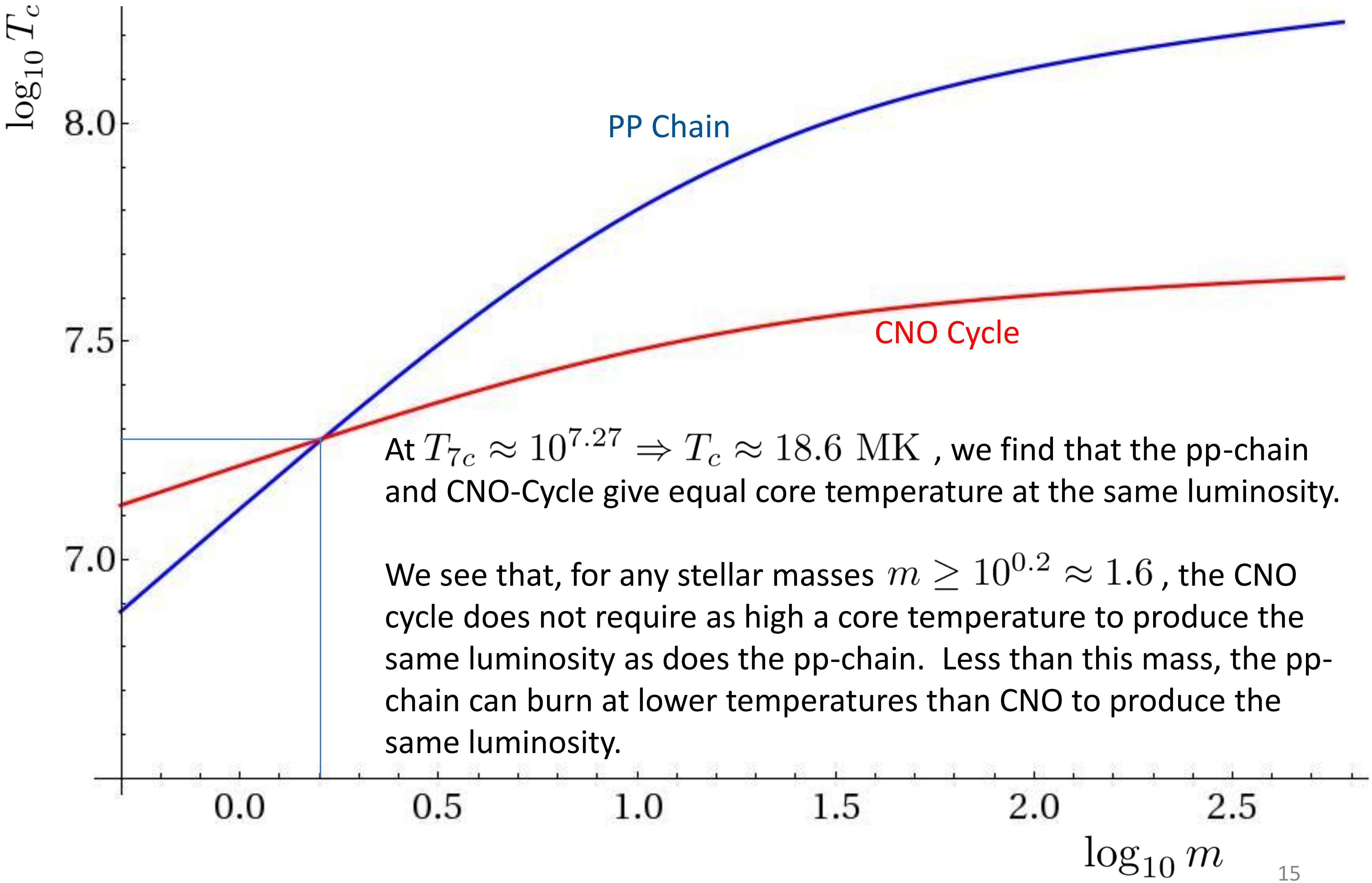
Mass-Luminosity: Main Sequence



We use this curve for the left hand sides of the equations on the previous page.

Data are from: G. Torres et al., Astron. Astrophys. Rev. (2009)

Main Sequence: Core Temperature vs Mass (for fixed Luminosity)



At $T_{7c} \approx 10^{7.27} \Rightarrow T_c \approx 18.6\ \mathrm{MK}$, we find that the pp-chain and CNO-Cycle give equal core temperature at the same luminosity.

We see that, for any stellar masses $m \geq 10^{0.2} \approx 1.6$, the CNO cycle does not require as high a core temperature to produce the same luminosity as does the pp-chain. Less than this mass, the pp-chain can burn at lower temperatures than CNO to produce the same luminosity.

A Consistency Check on the Crossover Temperature

At the crossover temperature, the energy generation rates of both burning cycles must be equal. So, set $\epsilon_{pp} = \epsilon_{CNO}$

$$\epsilon_{pp} = \rho X^2 \epsilon_{0_{pp}} \left(\frac{T}{T_0}\right)^{n_{pp}} = \rho X X_{CNO} \epsilon_{0_{CNO}} \left(\frac{T_6}{25}\right)^{n_{CNO}} = \epsilon_{CNO}$$

$$\Rightarrow F \frac{\epsilon_0^{pp} X^2}{(\mu\beta)^3 \; m^2} T_{7c}^{n_{pp}+3} = F \frac{\epsilon_0^{CNO} X X_{CNO}}{(\mu\beta)^3 \; m^2} \left(\frac{10}{25}\right)^{n_{CNO}} T_{7c}^{n_{CNO}+3}$$

For you to show this result for ϵ_{CNO} from the equations on page 5 and 6

Many terms cancel on both sides, and we require the temperature to be the same. After cancelling density, mass, $\mu\beta$ and so on, and solving for the common temperature:

$$T_{7c} = \left[\frac{\epsilon_0^{pp}}{\epsilon_0^{CNO}} \frac{X}{X_{CNO}} \left(\frac{25}{10}\right)^{n_{CNO}} \right]^{1/(n_{CNO} - n_{pp})} \approx 1.72$$

Luminosity vs Effective Temperature: Hertzsprung-Russel Diagram

The previous plot tells us that: For a particular value of luminosity, the pp-chain requires a higher core temperature than does the CNO cycle for stellar masses $m \geq 1.6$.

We cannot, of course, measure the core temperature of stars, so the previous result would otherwise be only purely theoretical.

However, astronomers do measure the effective surface temperatures of stars. Our Main Sequence model should be able to at least quantitatively show the trend of stellar surface temperatures to confirm the previous theoretical result of how the two reaction cycles depend on core temperature and stellar mass.

In other words: can the stellar surface temperatures be used to definitively determine where in the Main Sequence stars the pp-chain dominates, and where the CNO-cycle dominates?

Let's work toward the answer to this. Let the data tell us how the nuclear reactions in the core are connected to the temperature and luminosity we observe from the stars.

Back on page 28, we had: $\rho_c = F\frac{T_{7c}^3}{(\mu\beta)^3\ m^2} \Rightarrow T_{7c} = \frac{m^{2/3}\mu\beta}{F^{1/3}}\rho_c^{1/3}$

Now, the surface temperature of the star is related to its luminosity by the Black Body relationship, $L_* = 4\pi R_*^2 \sigma T_e^4$ with Stefan-Boltzmann constant:

$$\sigma = 5.67 \times 10^{-5}\ \mathrm{erg\ cm^{-2}\ s^{-1}\ K^{-4}}$$

This is how we will get the effective temperature related to the nuclear luminosity.

We need to relate the core temperature above to the stellar radius. We can do this with the Polytrope formalism because $\frac{\bar{\rho}}{\rho_c} = -3\left(\frac{1}{\xi}\frac{d\phi}{d\xi}\right)_{\xi_*} \approx \frac{1}{54}$ and $\bar{\rho} = \frac{3M_*}{4\pi R_*^3}$

Using: $\frac{M_\odot}{R_\odot^3} = 5.93$ $\Rightarrow \rho_c = 54\frac{3M_*}{4\pi R_*^3} \approx 76.5\frac{m}{r^3}$

We use this now in the equation up top to express the core temperature in terms of the stellar radius (and mass). Using $F = 10.27$, gives the simple result:

$$\boxed{T_{7c} = 1.95\mu\beta\frac{m}{r}}$$

Coupled to each other by structure constraints

We have a first "hint" here of things to come from the last equation:

$$\boxed{T_{7c} = 1.95\mu\beta\frac{m}{r}}$$ (Core temp related to radius)

As stellar mass grows, we expect the core temperature to increase, which would seem to suggest that the pp-burning will dominate over CNO burning. However, as the core temperature increases, the star's **structure** will adopt a new configuration; namely, the radius will grow (expansion) and it will be the competition of the mass versus the extent of expansion (growing r) that will regulate the core temperature and, therefore, the type of nuclear burning happening within the core.

Let us continue. At the stellar surface: $L_* = 4\pi R_\odot^2 r^2 \sigma T_e^4$

$$\boxed{\Rightarrow r = \frac{\sqrt{L_*/L_\odot}}{\sqrt{4\pi\sigma}\ R_\odot T_e^2}\ L_\odot^{1/2} = 3.34\times 10^7\left(\frac{L_*}{L_\odot}\right)^{1/2} T_e^{-2}}$$

Sub into equation up top: $$\boxed{T_{7c} = 5.84\times 10^{-8}\frac{\mu\beta m}{\sqrt{L/L_\odot}}T_e^2}$$

Almost done. On page 32, we derived, for the pp-chain, the following result:

$$\frac{L_*^{pp}}{L_\odot} = 1.18 \frac{X^2}{m(\mu\beta)^3\ (3u+n)^{3/2}} T_{7c}^{n+3}$$

Sub in the last expression for T_{7c} , and after some algebra, we finally have:

$$\left(\frac{L_*^{pp}}{L_\odot}\right)^{\frac{n+5}{2}} = \frac{1.18X^2}{(3u+n)^{3/2}} (\mu\beta)^n\ m^{n+2}\ T_e^{2(n+3)} \qquad n = 4.6$$

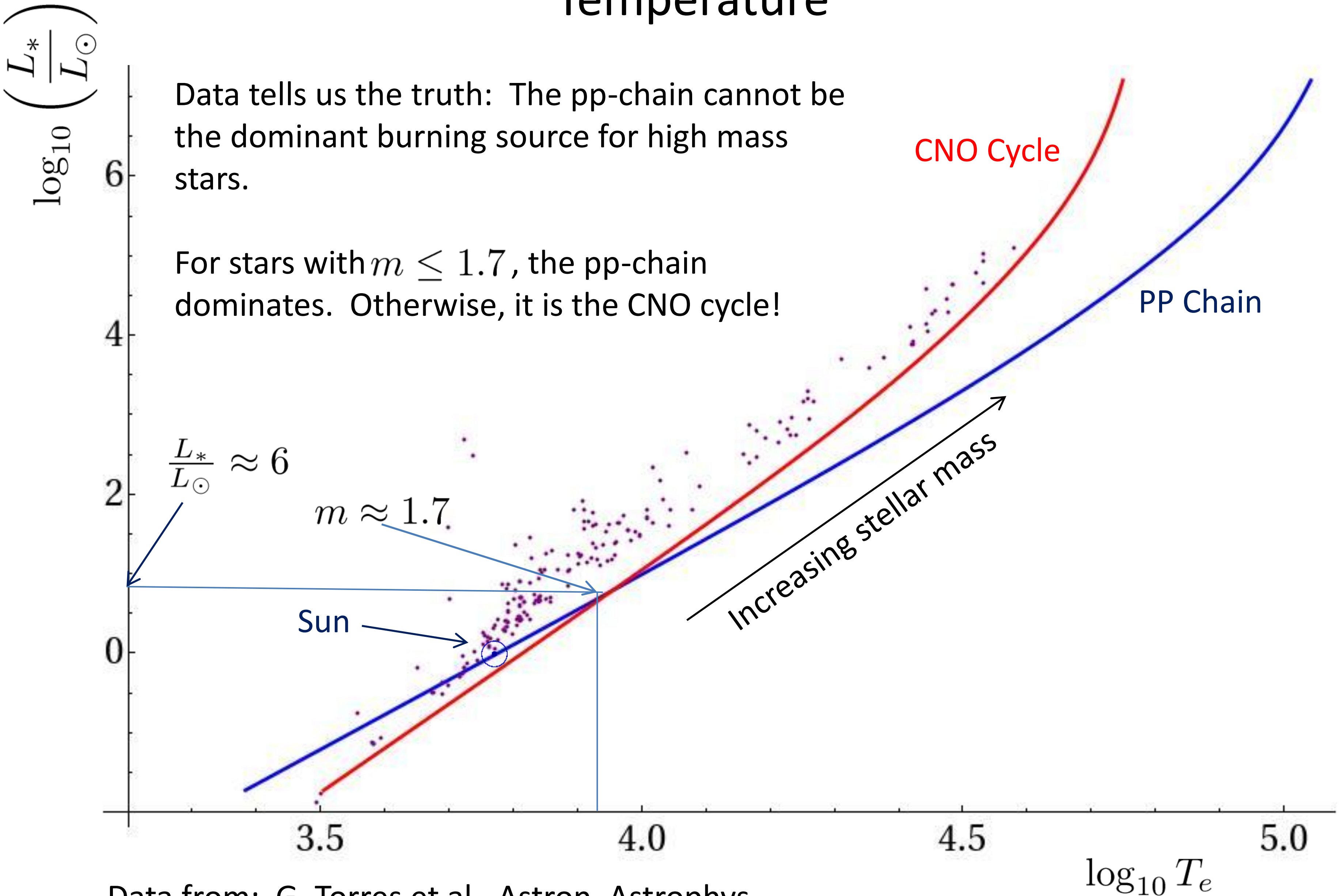
$$\left(\frac{L_*^{CNO}}{L_\odot}\right)^{\frac{n+5}{2}} = \frac{3.79 \times 10^5 X X_{CNO}}{(3u+n)^{3/2}} \left(\frac{10}{25}\right)^n\ (\mu\beta)^n\ m^{n+2}\ T_e^{2(n+3)} \qquad n = 16.7$$

So, we choose a value for m, use EQE to get β and use the mass-luminosity function of Lecture 5 on the LHS. Finally, solve for T_e as a function of stellar luminosity. Result on next slide, with Main Sequence data.

Main Sequence: Luminosity vs Effective Surface Temperature

Data tells us the truth: The pp-chain cannot be the dominant burning source for high mass stars.

For stars with $m \leq 1.7$, the pp-chain dominates. Otherwise, it is the CNO cycle!

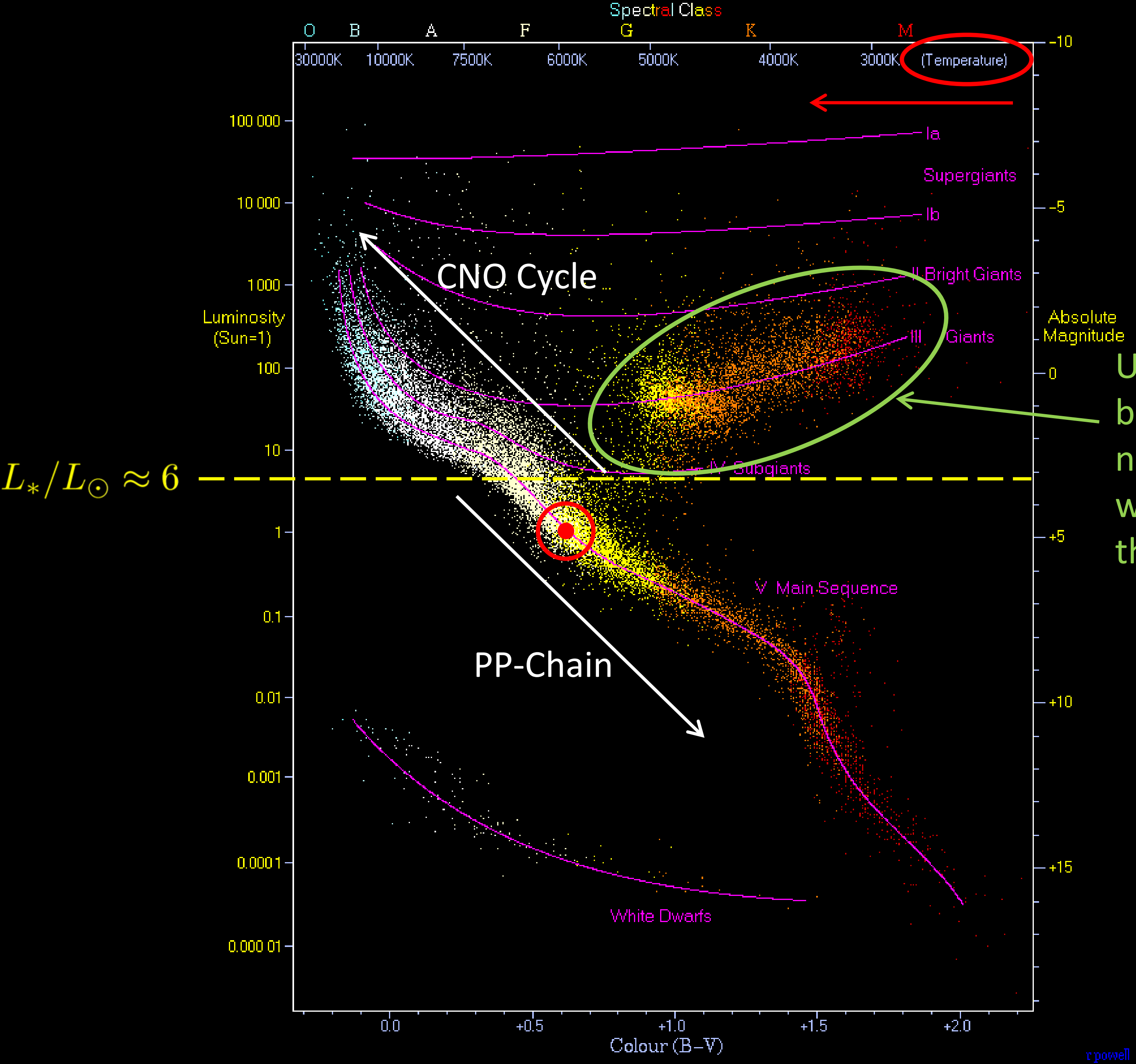


Data from: G. Torres et al., Astron. Astrophys. Rev. (2009)

Summary: Main Sequence Stars

- We have learned the mass-luminosity relationship based on the physics of the stellar structure equations + a “simple” Polytrope model
- Using the mass-luminosity relationship (data), and nuclear reaction results (to be taught in the New Year), we have extended the above result to learn the behaviour of stellar core temperatures as a function of stellar mass (page 12)
- For high mass stars, the pp-chain requires a higher core temp. than the CNO cycle to produce the same luminosity. (page 12)
- Our Polytrope Main Sequence model, while not perfect, agrees with the **trend** of the observational Luminosity-Effective Temperature data. The data and model show that, for those stars with $m \geq 1.6$ the luminosity is the result of the CNO nuclear reaction cycle; pp-chain nuclear reaction cycle is primary stellar energy source for masses less than ≈ 1.6 .

Hertzprung-Russel Diagram



Understanding how stars become Giants, and the nuclear processes within, will be our next step in the New Year.