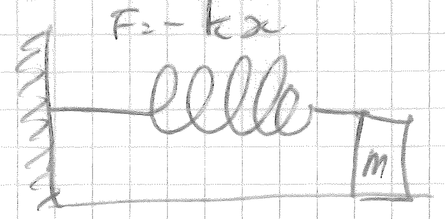
Spring

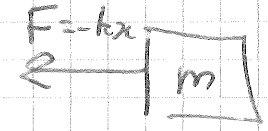


$$F_s = -kx$$

$$-\frac{dU}{dx} = F \Rightarrow \int_0^x dU = \int_0^x -F\,dx = \int_0^x kx\,dx$$

$$\Rightarrow U(x) - \cancelto{0}{U(0)} = \frac{kx^2}{2}$$

$$\boxed{U(x) = \frac{1}{2}kx^2}$$



Newton: $m\ddot{x} = -kx$, with $x(0) = x_0$, $\dot{x}(0) = 0$

$$\Rightarrow \ddot{x} + \frac{k}{m}x = 0$$

Trial Solution: $x(t) = A\cos(\omega t + \phi)$

then $\dot{x} = -A\omega\sin(\omega t + \phi)$

$$\ddot{x} = -A\omega^2\cos(\omega t + \phi)$$

Put into EOM: $-\cancel{m}A\omega^2\cos(\omega t + \phi) + \frac{k}{m}A\cos(\omega t + \phi) = 0$

$$\Rightarrow \boxed{\omega = \sqrt{k/m}}$$

$x(0) = \cancel{x_0} = A\cos(\omega t + \phi) \Rightarrow \phi = 0$, $A = x_0$

$\dot{x}(0) = 0 = -A\omega\sin 0$ ✓

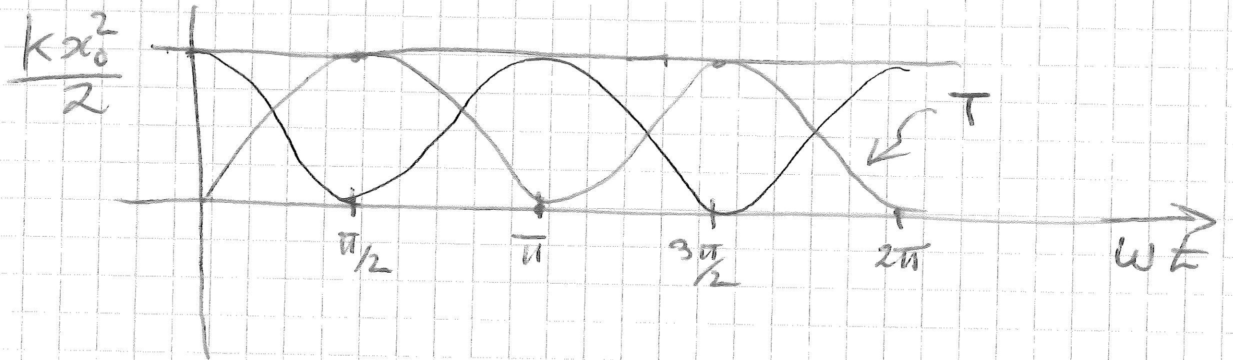
Solution: $x(t) = x_0 \cos\left(\sqrt{k/m}\; t\right)$

Now, total energy of system:

$$E = T + U = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$$

$$= \frac{1}{2}\left\{ \cancel{m} x_0^2 \frac{k}{\cancel{m}} \sin^2(\omega t) + k x_0^2 \cos^2(\omega t) \right\}$$

$$= \frac{1}{2} k x_0^2$$



Example (3D)

$$V(r) = \alpha x^2 + \beta x y + \gamma z + C$$

a) What is $\vec{F}$?

$$\vec{F} = -\nabla V = -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$

$$= -\left(2\alpha x + \beta y,\ \beta x,\ \gamma\right)$$

b) At $t=0$, particle passes through origin with <u>speed</u> v_0. What is speed at $\vec{r} = (1,2,1)$?

Use energy theorem:

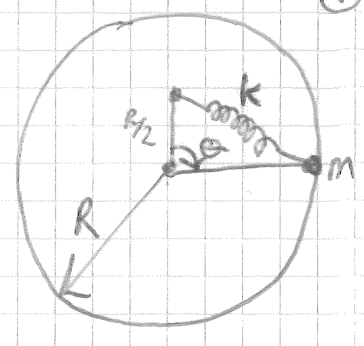
$$E = E(0,0,0) = \frac{1}{2} m v_0^2 + C$$

$$= E\,(\text{anywhere}) = \frac{1}{2} m v^2 + \alpha + 2\beta + \gamma + C$$

$$\Rightarrow \frac{1}{2} m v^2 = \frac{1}{2} m v_0^2 - \alpha - 2\beta - \gamma$$

$$v^2 = v_0^2 - \frac{2}{m}(\alpha + 2\beta + \gamma)$$

When bead is at <u>top of ring</u>, spring is <u>not extended</u>



a) Find $V(\theta)$, as measured from center of circle.

Natural spring length for $F(x)=0 \Rightarrow l_s = R/2$.

$$L_s^2 = (R/2)^2 + R^2 - 2\frac{R^2}{2}\cos\theta$$

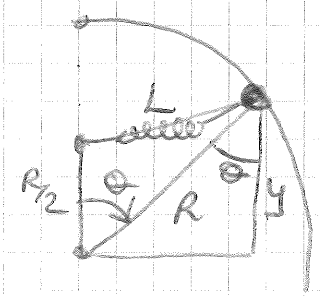
$$= R^2\left(\frac{5}{4} - \cos\theta\right) \Rightarrow L_s = R\sqrt{5/4 - \cos\theta}$$

$$= \frac{R}{2}\sqrt{5-4\cos\theta}$$

~~The potential E of spring is~~ $\cancel{U_s(\theta) = -\frac{k}{2}\left(\frac{R^2}{4}(5-4\cos\theta) - \frac{R^2}{4}\right)}$

$$\cancel{\Rightarrow U_s(\theta) = -\frac{kR^2}{8}\cdot 4(1-\cos\theta) = -\frac{kR^2}{2}(1-\cos\theta)}$$

<u>Gravitational Energy</u>:



$$y = R\cos\theta$$

$$U_g(\theta) = mgy = mgR\cos\theta$$

$$\cancel{U_{Total} = mgR\cos\theta - \frac{kR^2}{2}(1-\cos\theta)}$$

Spring Potential energy $U_s(\theta)$

$$U_s(\theta) = \frac{1}{2}k\left(\frac{R}{2}\sqrt{5-4\cos\theta} - R/2\right)^2 = \frac{kR^2}{8}\left(\sqrt{5-4\cos\theta} - 1\right)^2$$

$$U_{Total}(\theta) = \frac{kR^2}{8}\left(\sqrt{5-4\cos\theta} - 1\right)^2 + mgR\cos\theta$$

b) What min KE required to go all the way around the loop?

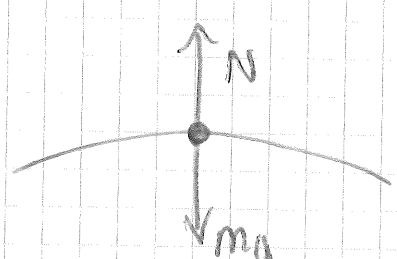
We use energy theorem: $E_{tot} = E_k + V$

We require velocity at bottom of ring $v = 0$.

then: $E_{tot} = E_k + mgR = \frac{kR^2}{8}\left(\sqrt{5 - 4\cos\pi} - 1\right)^2 + mgR\cos\pi$

$$\Rightarrow E_k = -2mgR + \frac{kR^2}{8} \cdot 4 = \frac{kR^2}{2} - 2mgR$$

c) What is force at top and bottom for the KE in (b)?



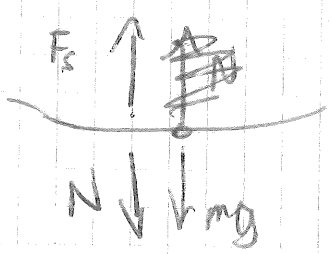
Top:

$$N - mg = -\frac{mv^2}{R} = -\frac{2}{R}E_k$$

$$N = -\frac{2}{R}\left\{\frac{kR^2}{2} - 2mgR\right\} + mg \cancel{+kR - 5mg}$$

$$\Rightarrow \boxed{N = -kR + 5mg}$$

Bottom:



$$F_s + N - mg = \frac{mv^2}{R} = 0$$

$$N = -mg + F_s$$

$$\Rightarrow \boxed{N = -mg + kR}$$

$\vec{P}$ before collision = $\vec{P}$ after collision

$$\vec{P}_i = \vec{P}_f$$

$$\Rightarrow m v_m = (M+m) v_f$$

$$\Rightarrow \boxed{v_f = \frac{m}{M+m} v_m} \quad (*)$$

Next, how far does combined system slide?

Can use ① constant acceleration or

② Work theorem $\Delta W = \Delta E_k$

$$\Delta E_k = E_{k_f} - E_{k_i} = \int_0^x F_f \cdot dx$$

$$\Rightarrow -\frac{1}{2}(M+m) v_f^2 = -\mu (M+m) g \int_0^x dx$$

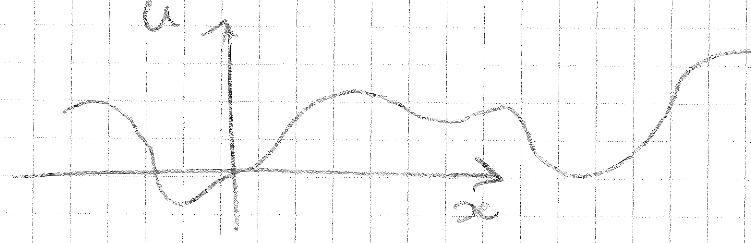
$$= -\mu g x (M+m)$$

$$\Rightarrow \boxed{x = \frac{v_f^2}{2\mu g} = \frac{m^2}{2(M+m)^2} \frac{v_m^2}{\mu g}}$$

Stability: We have $\vec{F} = -\nabla f(x,y,z)$.

Take 1-D case:

$$F_x = -\frac{dU}{dx}$$

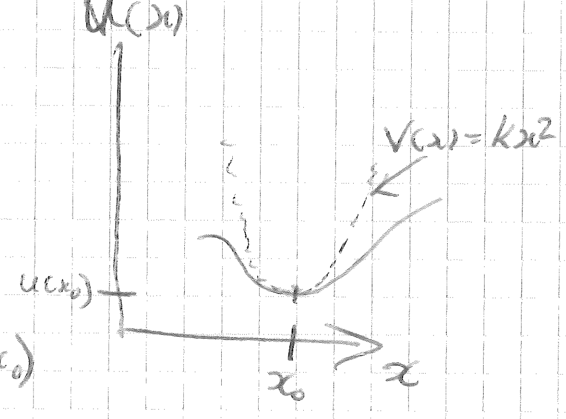


Where is $F_x = 0 \Rightarrow \frac{dU}{dx} = 0$ so look for extrema.

Next: is extrema a ~~max~~ minimum or maximum?

$$\frac{d^2U}{dx^2} < 0 \Rightarrow \text{maximum}$$

$$\frac{d^2U}{dx^2} > 0 \Rightarrow \text{minimum.}$$



Locally, around the minimum we can approximate the potential function ~~$U(x) = kx^2$~~

$$U(x) = \frac{k}{2}(x-x_0)^2 + U(x_0)$$

This is like the SHO potential.

$$F_x = -\frac{dU}{dx} = -k(x-x_0)$$

Momentum & Impulse

Newton's 2nd Law $\vec{F} = \frac{d\vec{p}}{dt}$

$$\Rightarrow \int d\vec{p} = \int \vec{F}\, dt = \Delta \vec{p} = \vec{p}_f - \vec{p}_i = \vec{I}$$

$$\boxed{\vec{I} \equiv \int \vec{F}\, dt = \vec{p}_f - \vec{p}_i}$$

"Impulse of a force equals change of momentum of the particle".

Example:

Golf ball struck by club and hit a distance d. Mass of ball is m.
What is impulse delivered to ball?

$$I = \int \vec{F} dt = \Delta \vec{p}$$

$\boxed{v_i = 0}$ before collision

Range of Projectile: $R = \frac{v_0^2}{g} \sin 2\theta_0 = d.$

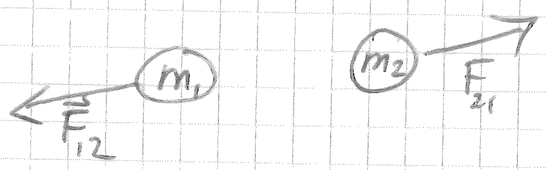
Use $\theta_0 = 45^\circ$ for max. distance

Then $v_f^2 = gd \Rightarrow v_f = \sqrt{gd}$

$$\Delta \vec{p} = (v_f - v_i)m = m v_f = \boxed{m\sqrt{gd} = I}$$

Conservation of Momentum (2 particle system)

We have $\vec{F}_{12} = -\vec{F}_{21}$



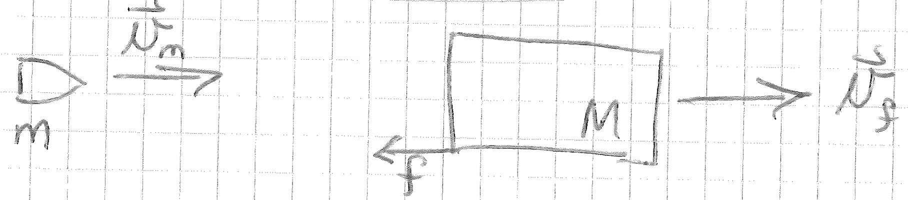
$$\Rightarrow \vec{F}_{12} + \vec{F}_{21} = 0$$

$$\Rightarrow \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = \frac{d}{dt}\left\{\vec{p}_1 + \vec{p}_2\right\} = \frac{d\vec{P}}{dt} = 0$$

When no <u>external</u> forces act on system.

$$\Rightarrow \vec{P} = \text{constant} = \vec{p}_1 + \vec{p}_2$$

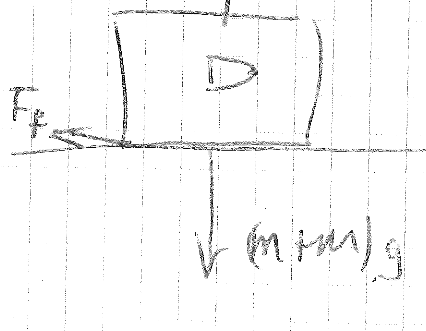
<u>Example</u>:



Bullet fired into block of mass M.
Given: m = bullet mass
v_m = initial bullet velocity
μ = coefficient of friction

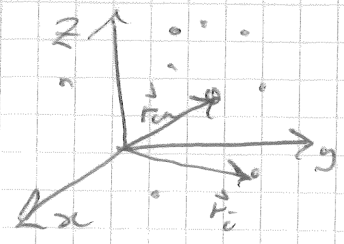
How far does wood + bullet travel before stopping?

$$\vec{F}_f = \mu N$$
$$= \mu g (m + M)$$



System of Particles

Define Center of Mass:



$$\vec{r}_{cm} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n}{\sum_i m_i} = \frac{\sum_i m_i\vec{r}_i}{M}$$

where $M = \sum_i m_i$

or $\vec{r}_{cm} = \frac{1}{M}\left(\sum_i m_i x_i, \sum_i m_i y_i, \sum_i m_i z_i\right)$

Total momentum of system:

$$\vec{P} = \sum_i \vec{p}_i = \sum_i m_i\vec{v}_i = M\vec{v}_{cm}$$

Because $\vec{v}_{cm} = \dot{\vec{r}}_{cm} = \frac{\sum_i m_i \dot{\vec{r}}_i}{M} = \frac{1}{M}\sum_i m_i\vec{v}_i$

Suppose system now has <u>external</u> forces acting on it. Each particle "i" has a force $\vec{F}_i$ from outside system acting on it. Also, within the system, between particles there are internal forces $\vec{F}_{ij}$.

Equation of motion for <u>one</u> particle is:

$$\vec{F}_i + \sum_{j=1}^{n}\vec{F}_{ij} = m_i\ddot{\vec{r}}_i = \dot{\vec{P}}_i$$

Now sum last equation over all particles in system:

$$\sum_{i=1}^{n} \vec{F}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \vec{F}_{ij} = \sum_{i=1}^{n} \dot{\vec{P}}_i$$

The double sum is zero. To see this, expand it and use Newton's 3rd law, $\vec{F}_{ij} = -\vec{F}_{ji}$ and $F_{ii} = 0$.

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \vec{F}_{ij} = \begin{aligned} &\left(0 + \vec{F}_{12} + \vec{F}_{13} + \dots + \vec{F}_{1n-1} + \vec{F}_{1n}\right) \\ &+ \left(\vec{F}_{21} + 0 + \vec{F}_{23} + \dots + \vec{F}_{2n-1} + \vec{F}_{2n}\right) \\ &+ \left(\vec{F}_{31} + \vec{F}_{32} + 0 + \dots + \vec{F}_{3n-1} + \vec{F}_{3n}\right) \\ &+ \dots \\ &\quad\vdots \\ &+ \left(\vec{F}_{n1} + \vec{F}_{n2} + \vec{F}_{n3} + \dots + \vec{F}_{nn-1} + 0\right) \end{aligned}$$

Sum each row from top with corresponding column from left, and all cancels.

$\therefore$, we have: $\sum_{i=1}^{n} \vec{F}_i = \dot{\vec{P}} = M\vec{a}_{cm}$ $\quad (\dot{\vec{P}} = \sum \dot{\vec{P}}_i)$

$\therefore$, the acceleration of the CoM of a particle system is the same as that of a single particle having a mass $M = \sum_i m_i$ acted upon by sum of all external forces.

When the external forces are zero (no forces at all) or when $\sum \vec{F}_i = 0$, then

$$\dot{\vec{P}} = 0 \Rightarrow \underline{\vec{P} = \text{constant}}$$

This is conservation of linear momentum

Example: A rocket in flight, mass m, breaks into 3 fragments of equal mass, $m/3$.
One fragment continues with $\vec{v}_0/2$, where $\vec{v}_0$ is original velocity of rocket just before breakup. The other 2 fragments go off at right angles to each other with same speeds.

Find: initial speeds of those other two fragments in terms of v_0.

~~Soltt~~ Solution.
At point of breakup, Conservation of linear momentum is:

$$m\vec{v}_{cm} = m\vec{v}_0$$

This must equal momentum of outgoing fragments:

$$\Rightarrow \quad m\vec{v}_0 = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3$$

$$= \frac{m}{3}(\vec{v}_1 + \vec{v}_2 + \vec{v}_3)$$

We had $\vec{v}_1 = \vec{v}_0/2$, so

$$\frac{5}{2}\vec{v}_0 = \vec{v}_2 + \vec{v}_3$$

Take dot product of both sides with themselves:

$$\frac{5}{2}\vec{v}_0 \cdot \frac{5}{2}\vec{v}_0 = (\vec{v}_2 + \vec{v}_3)\cdot(\vec{v}_2 + \vec{v}_3)$$

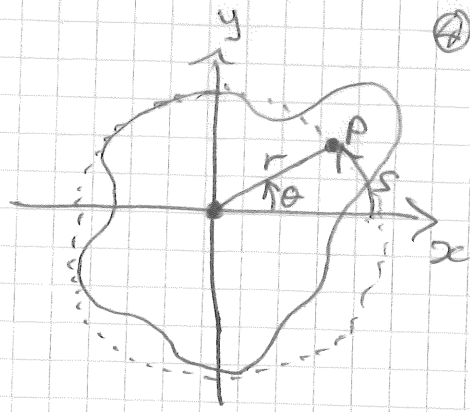
$$\Rightarrow \quad \frac{25}{4}v_0^2 = v_2^2 + v_3^2 + 2\underbrace{\vec{v}_2\cdot\vec{v}_3}_{0}$$

$$\Rightarrow \quad \frac{25}{4}v_0^2 = 2v_2^2 \qquad (v_2 = v_3)$$

$$\Rightarrow \quad v_2 = v_3 = \frac{5}{2\sqrt{2}}v_0$$

Rotation of Rigid Bodies:

Circular displacement to point P



$$s = r\theta$$

Define: angular velocity

$$\omega = \frac{d\theta}{dt} = \dot{\theta}$$

~~Angular momentum~~

Angular acceleration

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

When α is constant, then we have the usual kinematics equations:

① $\omega = \omega_0 + \alpha t$

② $\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$

③ $\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$ (comes from $\Delta E_k = \int F \cdot ds = \int m\alpha \cdot ds$)

Since $s = r\theta$, $v = \frac{ds}{dt} = r\dot{\theta} = r\omega$

and $a = r\ddot{\theta} = r\alpha$

Angular Momentum & Kinetic Energy of a System

For a single particle, we have

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\Rightarrow (\vec{r}\times \ldots) \qquad \vec{r}\times\vec{F} = \vec{r}\times\frac{d\vec{p}}{dt}$$

Consider $\frac{d}{dt}\{\vec{r}\times\vec{p}\} = \dot{\vec{r}}\times\vec{p} + \vec{r}\times\dot{\vec{p}}$

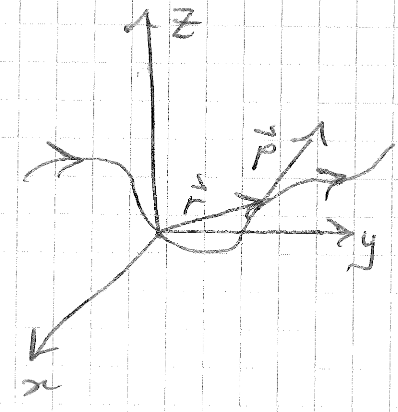
$$= \vec{v}\times m\vec{v} + \vec{r}\times\dot{\vec{p}}$$

$$= \vec{r}\times\dot{\vec{p}}$$

$\therefore \quad \vec{r}\times\vec{F} = \frac{d}{dt}\{\vec{r}\times\vec{p}\}$

And we define $\vec{r}\times\vec{p} = \vec{L}$ = angular momentum.

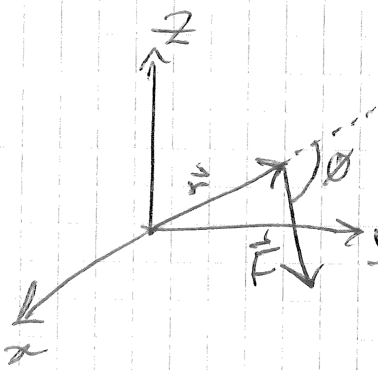
$\vec{L}$ is a vector that lies in a plane perpendicular to $\vec{r}$ and $\vec{p}$



We also call the quantity

$\vec{r}\times\vec{F} = \vec{M}$ = torque

$\therefore \quad \vec{M} = \frac{d\vec{L}}{dt}$ for single particle



System of Particles:

Define $\vec{L} = \sum_{i=1}^{n} (\vec{r}_i \times m_i \vec{v}_i)$

Now take the time derivative

$$\frac{d\vec{L}}{dt} = \sum_i^n (\vec{v}_i \times m_i \vec{v}_i) + \sum_i^n (\vec{r}_i \times m_i \vec{a}_i)$$

$$= \sum_i^n (\vec{r}_i \times m_i \vec{a}_i)$$

Now $m_i \vec{a}_i = \vec{F}_i + \sum_{j=1}^{n} \vec{F}_{ij}$

So, $\frac{d\vec{L}}{dt} = \sum_i^n \left(\vec{r}_i \times \left(\vec{F}_i + \sum_i^n \vec{F}_{ij}\right)\right)$

$$= \sum_i^n (\vec{r}_i \times \vec{F}_i) + \sum_i^n \sum_j^n (\vec{r}_i \times \vec{F}_{ij})$$

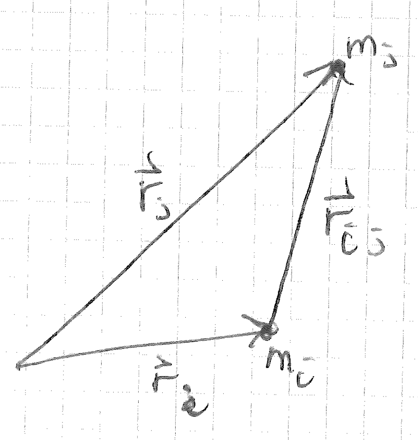
The double sum will have pairs of terms of the form $(\vec{r}_i \times \vec{F}_{ij}) + (\vec{r}_j \times \vec{F}_{ji})$ (*) (similar to other double sum)

Now $\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$, and $\vec{F}_{ij} = -\vec{F}_{ji}$

Then (*) $\Rightarrow$ $\vec{r}_i \times \vec{F}_{ij} + (\vec{r}_{ij} + \vec{r}_i) \times (-\vec{F}_{ij})$

$= \vec{r}_i \times \vec{F}_{ij} - \vec{r}_i \times \vec{F}_{ij} - \vec{r}_{ij} \times F_{ij}$

$= -\vec{r}_{ij} \times \vec{F}_{ij}$



$\Rightarrow$ If $\vec{F}_{ij}$ acts along $\vec{r}_{ij}$ then the double sum is zero

Finally, $\dfrac{d\vec{L}}{dt} = \sum_{i=1}^{n} (\vec{r}_i \times \vec{F}_i) \equiv \vec{M}$

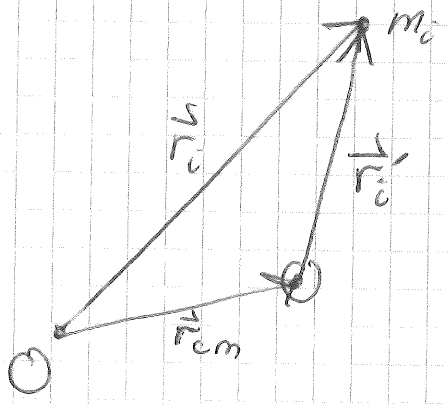
<u>Result</u>: time rate of change of angular momentum of a <u>system</u> is equal to total moment (torque) of external forces acting on system.

If there are no external forces, then

$$\frac{d\vec{L}}{dt} = \vec{M} = 0$$

$\Rightarrow \vec{L} = \text{constant}$ $\boxed{\text{Conservation of Angular Momentum}}$

Alternative Expression for $\vec{L}$:



In term of the CM, we can write

$\vec{r}_i = \vec{r}_{cm} + \vec{r}_i'$

$\Rightarrow \vec{v}_i = \vec{v}_{cm} + \vec{v}_i'$

Then $\vec{L} = \sum_i (\vec{r}_i \times m \vec{v}_i)$

$$= \sum_i (\vec{r}_{cm} + \vec{r}_i') \times m_i(\vec{v}_{cm} \times \vec{v}_i')$$

$$= \sum_i (\vec{r}_{cm} \times m_i \vec{v}_{cm}) + \sum_i (\vec{r}_{cm} \times m_i \vec{v}_i')$$

$$+ \sum_i (\vec{r}_i' \times m_i \vec{v}_{cm}) + \sum_i (\vec{r}_i' \times m_i \vec{v}_i')$$

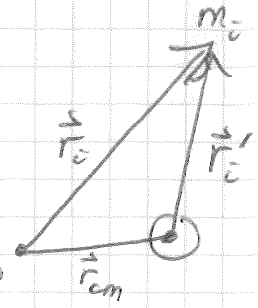
$$= \vec{r}_{cm} \times \Big(\sum_i m_i\Big) \vec{v}_{cm} + \vec{r}_{cm} \times \Big(\sum_i m_i \vec{v}_i'\Big)$$

$$+ \Big(\sum_i m_i \vec{r}_i'\Big) \times \vec{v}_{cm} + \sum_i (\vec{r}_i' \times m_i \vec{v}_i')$$

$$= \vec{r}_{cm} \times M \vec{v}_{cm} + \sum_i \vec{r}_i' \times m_i \vec{v}_i'$$

General Theorem of Angular Momentum.

We had, in the most general case of a system of particles, with internal forces:



$$\frac{d\vec{L}}{dt} = \vec{M}$$

or:
$$\frac{d}{dt}\left(\sum_i \vec{r}_i \times m_i \vec{v}_i\right) = \sum_i \vec{r}_i \times \vec{F}_i$$

with $\vec{F}_i$ the <u>external</u> force acting on particle i.

From vector diagram, we have $\vec{r}_i = \vec{r}_{cm} + \vec{r}_i'$

Sub into ~~LHS~~ top equation:

$$\frac{d}{dt}\left[\sum_i m_i(\vec{r}_{cm} + \vec{r}_i') \times (\vec{v}_{cm} + \vec{v}_i')\right] = \sum_i (\vec{r}_{cm} + \vec{r}_i') \times \vec{F}_i$$

$$\Rightarrow \frac{d}{dt}\left[\vec{r}_{cm} \times \vec{v}_{cm} M + \vec{r}_{cm} \times \underbrace{\sum_i m_i \vec{v}_i'}_{=0} + \vec{v}_{cm} \times \underbrace{\sum_i m_i \vec{r}_i'}_{=0} + \sum_i m_i \vec{r}_i' \times \vec{v}_i'\right]$$

$$= M \vec{r}_{cm} \times \vec{a}_{cm} + \frac{d}{dt}\sum_i m_i \vec{r}_i' \times \vec{v}_i' = \vec{r}_{cm} \times \sum_i \vec{F}_i + \sum_i \vec{r}_i' \times \vec{F}_i$$

$$= \vec{r}_{cm} \times \sum_i m_i \vec{a}_i + \sum_i \vec{r}_i' \times \vec{F}_i$$

Last lecture we had (first result): $\sum_i \vec{F}_i = \sum_i m_i \vec{a}_i = M \vec{a}_{cm}$

$\therefore$ 1st term on LHS cancels 1st term on RHS:

So, finally, we have: $\frac{d}{dt}\sum_i m_i \vec{r}_i' \times \vec{v}_i' = \sum_i \vec{r}_i' \times \vec{F}_i$

$$\Rightarrow \boxed{\frac{d\vec{L}'}{dt} = \vec{M}'}$$

Change in angular momentum about the C of M is equal to total torque about the center of mass.

Kinetic Energy of the System

$$E_k = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i (\vec{v}_i \cdot \vec{v}_i)$$

Now, use $\vec{v}_i = \vec{v}_{cm} + \vec{v}_i'$

Then $E_k = \frac{1}{2} \sum_i m_i (\vec{v}_{cm} + \vec{v}_i') \cdot (\vec{v}_{cm} + \vec{v}_i')$

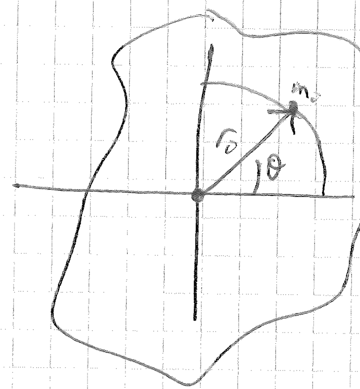
$$= \frac{1}{2} \sum_i m_i v_{cm}^2 + \sum_i m_i (\vec{v}_{cm} \cdot \vec{v}_i') + \frac{1}{2} \sum_i m_i v_i'^2$$

$$= \frac{1}{2} v_{cm}^2 \sum_i m_i + \underbrace{\vec{v}_{cm} \cdot \sum_i m_i \vec{v}_i'}_{= 0 \quad (\vec{v}_i = \vec{v}_{cm} + \vec{v}_i')} + \frac{1}{2} \sum_i m_i v_i'^2$$

$$= \frac{M}{2} v_{cm}^2 + \frac{1}{2} \sum_i m_i v_i'^2$$

Total kinetic energy is the KE of translation plus KE of motion relative to center of mass.

Kinetic energy of Rotating Rigid Body



$$E_{k_i} = \frac{1}{2} m_i v_i^2$$

$$= \frac{1}{2} m_i r_i^2 \omega^2 \quad \left(\begin{array}{c}\text{all particles have} \\ \text{same } \omega\end{array}\right)$$

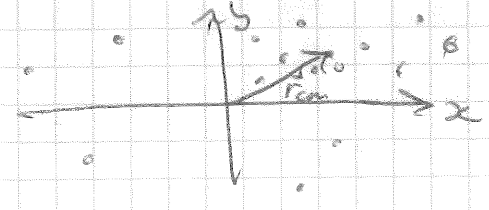
$$\therefore E_k = \sum_i E_{k_i} = \frac{1}{2} \omega^2 \sum_i m_i r_i^2$$

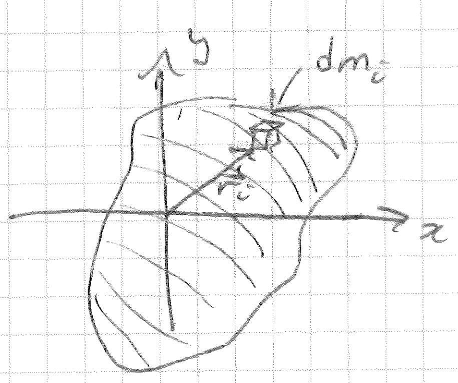
$$= \frac{1}{2} I \omega^2$$

Where $I = \sum_i m_i r_i^2$ = moment of inertia

Center of Mass

$$\vec{r}_{cm} = \frac{\sum_i m_i \vec{r}_i}{M}$$





For continuous matter distribution:

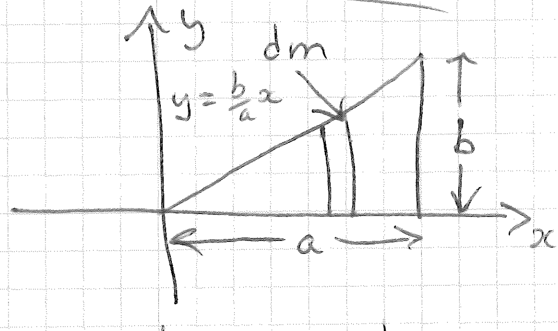
$$\vec{r}_{cm} = \frac{\int \vec{r}\, dm}{\int \rho\, dV}$$

$dm_i = \rho\, dV$

$$\Rightarrow x_{cm} = \frac{\int \rho x\, dV}{\int \rho\, dV}$$

$$y_{cm} = \frac{\int \rho y\, dV}{\int \rho\, dV}, \quad z_{cm} = \frac{\int \rho z\, dV}{\int \rho\, dV}$$

Examples: Right triangle, $\rho = \rho_0$



$dm = \rho_0\, y\, dx$

$$x_{cm} = \frac{1}{M} \int \rho x\, dA$$

$$= \frac{1}{M} \int_0^a \rho_0\, x\, y\, dx$$

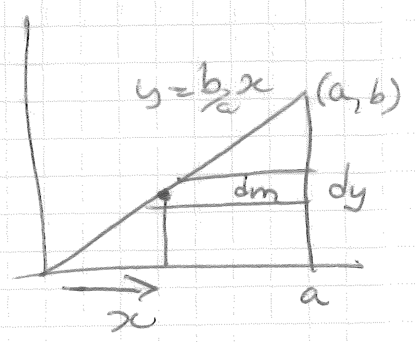
$$= \frac{1}{M} \rho_0 \cdot \frac{b}{a} \int_0^a x^2\, dx = \frac{\rho_0 b a^2}{3M}$$

$$= \frac{\rho_0 b a^2}{3 \rho_0 \frac{ab}{2}} = \boxed{\frac{2}{3} a = x_{cm}}$$

$$M = \rho_0 \int_0^a y\, dx = \rho_0 \frac{b}{a} \int_0^a x\, dx$$

$$= \frac{\rho_0 b}{2a} a^2 = \frac{\rho_0 ab}{2}$$

$$y_{cm} = \frac{1}{M}\rho_0 a \int_0^b y\left(1-\frac{y}{b}\right)dy$$



$$dm = \rho_0 (a-x)\,dy$$

$$= \rho_0 a \left(1 - \frac{y}{b}\right) dy$$

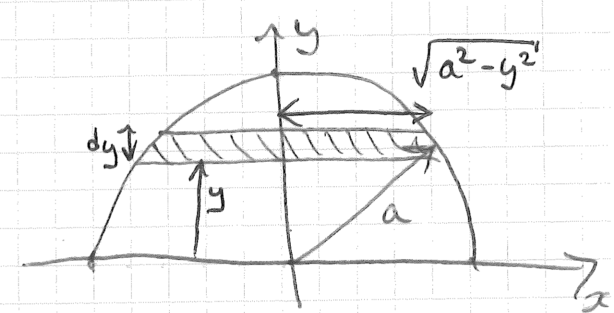
$$= \frac{2\rho_0 a}{ab\rho_0}\left\{\frac{y^2}{2} - \frac{y^3}{3b}\right\}\Big|_0^b$$

$$= \frac{2}{b}\frac{b^2}{6} = \boxed{\frac{b}{3} = y_{cm}}$$

$$\therefore\ \vec{r}_{cm} = \left(\frac{2}{3}a,\ \frac{b}{3}\right)$$

<u>Solid Hemisphere</u> (uniform density)

$$dm = \rho_0 \cdot \pi\left(\sqrt{a^2-y^2}\right)^2 dy$$



$$y_{cm} = \frac{1}{M}\int_0^a \pi\rho_0 (a^2-y^2)\,y\,dy$$

$$M = \rho_0 \int_0^a \pi (a^2-y^2)\,dy$$

$$= \frac{1}{M}\pi\rho_0\left\{\frac{a^2y^2}{2} - \frac{y^4}{4}\right\}_0^a$$

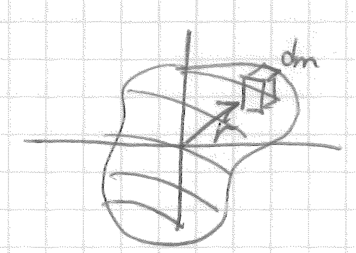
$$= \pi\rho_0\left\{a^2y - \frac{y^3}{3}\right\}\Big|_0^a$$

$$= \frac{3}{2a^3}\frac{a^4}{4} = \frac{3a}{8}$$

$$= \pi\rho_0 a^3 \cdot \frac{2}{3}$$

$$= \frac{2\pi\rho_0 a^3}{3}$$

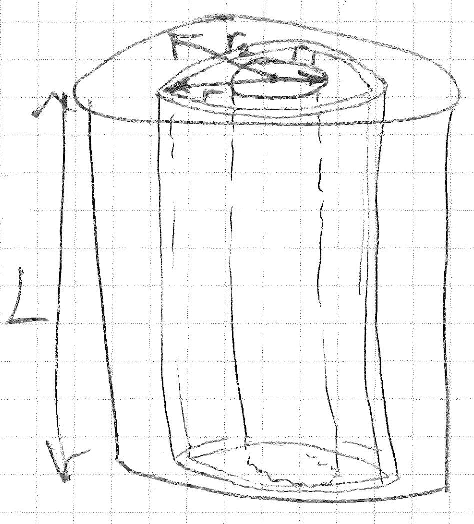
Moments of Inertia



$$I = \sum_i m_i r_i^2 \Rightarrow \int r^2 \, dm$$

Remember! the r here is the distance from dm to the axis of rotation.

Example: Hollow Cylinder



$$dm = 2\pi r L \, dr \, \rho_0$$

$$I = 2\pi L \rho_0 \int_{r_1}^{r_2} r^3 \, dr$$

$$= \frac{\pi L \rho_0}{2} (r_2^4 - r_1^4)$$

$$= \frac{\pi L \rho_0}{2} (r_2^2 - r_1^2)(r_2^2 + r_1^2)$$

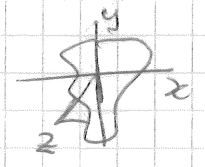
What is ρ_0?

$$\text{Volume} = \pi L (r_2^2 - r_1^2)$$

$$\Rightarrow \rho_0 = \frac{M}{\pi L (r_2^2 - r_1^2)}$$

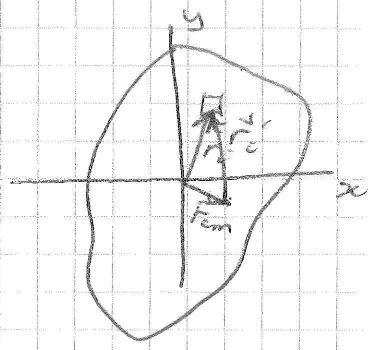
$$\therefore I = \frac{M}{2} (r_2^2 + r_1^2)$$

Parallel-Axis Theorem



$$I = \sum_i m_i r_i^2 = \sum_i m_i (x_i^2 + y_i^2)$$ (rotation about z)

We have $\vec{r}_i = \vec{r}_{cm} + \vec{r}_i'$



Then $I = \sum m_i \left[(x_{cm} + x_i')^2 + (y_{cm} + y_i')^2 \right]$

$$= \sum_i m_i (x_{cm}^2 + y_{cm}^2) + \sum_i m_i (x_i'^2 + y_i'^2)$$

$$+ 2 \underbrace{\sum_i m_i x_{cm} x_i'}_{0} + 2 \underbrace{\sum_i m_i y_{cm} y_i'}_{0}$$

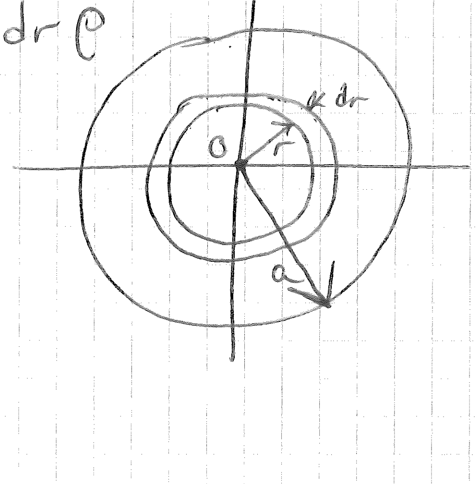
$$= M r_{cm}^2 + I_{cm}$$

$\therefore$ $\boxed{I = M r_{cm}^2 + I_{cm}}$

$$= M r_{cm}^2 + \sum_i m_i r_i'^2$$

Example: Disk

$dm = 2\pi r\, dr\, \rho$



$$I_0 = \int_0^a r^2 dm$$

$$\rho = \frac{M}{\pi a^2}$$

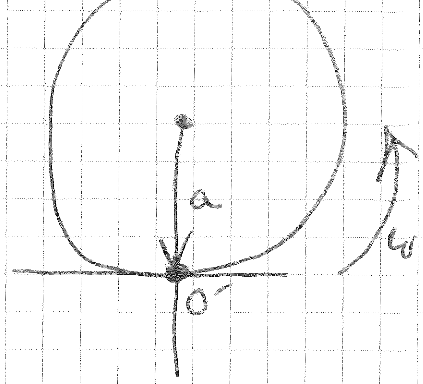
$$= 2\pi \rho \int_0^a r^3 dr$$

$$= \frac{2\pi \rho}{4} r^4 \Big|_0^a = \frac{2\pi}{4} \rho a^4$$

$$= \frac{\pi}{2\pi a^2} M a^4 = \frac{M a^2}{2}$$

Now rotate as shown.

What is $I_{0'}$?



Parallel axis theorem:

$$I_{0'} = I_{cm} + Ma^2$$

$$= \frac{Ma^2}{2} + Ma^2 = \frac{3}{2} Ma^2$$