Lecture – 4

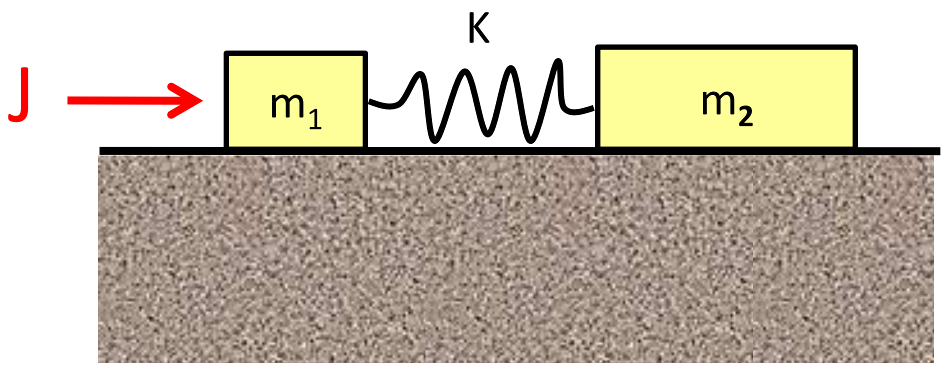
Torque and Levers
The Mechanics of Rigid Bodies

Experimentalphysik I in Englischer Sprache

13-11-08

Homework 3

Two bodies of masses m_1 and m_2 are free to move along a horizontal straight, frictionless track. They are connected by a spring with constant K.



The system is initially at rest before an instantaneous impulse **J** is give to m_1 along the direction of the track.

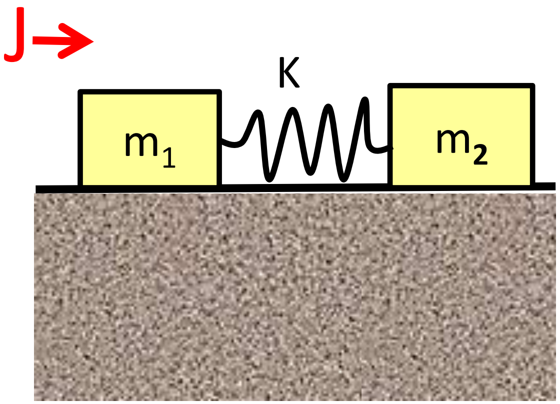
Q) Determine the motion of the system and find the energy of oscillation of the two bodies

A) You'll need to use ideas of energy, momentum conservation and derive the eqn of motion of 2 coupled masses

(i)CONSERVATION OF MOMENTUM and ENERGY

(ii) NATURAL FREQUENCY ω_0 OF A 2 BODY HARMONIC OSCILLATOR

We learnt last time that <u>impulse</u> $\vec{J}$ is the total change of the momentum $\vec{p}$ of a body:



<u>Conservation of momentum</u>

Since m_1 is initially at rest, it acquires a momentum $\vec{J}$ when it is given the instantaneous impulse $\vec{J}$

$$\Rightarrow \vec{J} = m_1\vec{v}_1$$, where $\vec{v}_1$ is the velocity of mass m_1 after the impact

This momentum is eventually "shared" by both bodies when their "center of mass" starts moving with a velocity V

Since momentum is conserved in a system where only internal forces are acting

$$\Leftrightarrow \quad m_1\vec{v}_1 = (m_1 + m_2)\vec{V} \quad , \text{or} \quad |\vec{V}| = \frac{|\vec{J}|}{m_1 + m_2} = \frac{|\vec{J}|}{M} \quad , \text{where } M = m_1 + m_2$$

<u>Conservation of Energy</u>

The previous equation gives the velocity of the center of mass of the system of two bodies:

$$\Rightarrow \text{TOTAL KINETIC ENERGY} \qquad E_t = \frac{1}{2}MV^2 = \frac{1}{2}M\left(\frac{J}{M}\right)^2 = \frac{J^2}{2M}$$

①

Now, this KE is changed to PE as the spring connecting the two masses gets compressed and stretched......

The energy "used up" in compressing the spring is the difference between the initial KE of mass m_1 $\left(= \frac{1}{2} m_1 v^2\right)$ and the final translational KE of the system $\left(= \frac{1}{2}(m_1+m_2)V^2\right)$

Representing this energy by H_0, we obtain $\Rightarrow H_0 = \frac{1}{2} m_1 v_1^2 - \frac{1}{2}(m_1+m_2)V^2$

$$\text{so,} \quad H_0 = \frac{1}{2} m_1 \left(\frac{J}{m_1}\right)^2 - \frac{1}{2}(m_1+m_2)\left[\frac{J}{(m_1+m_2)}\right]^2 \quad \text{since } V = \frac{J}{m_1+m_2} \quad v_1 = \frac{J}{m_1}$$

$$H_0 = \frac{J^2}{2m_1} - \frac{J^2}{2(m_1+m_2)} = \frac{J^2}{2}\left[\frac{1}{m_1} - \frac{1}{(m_1+m_2)}\right]$$

$$H_0 = \frac{J^2}{2}\left[\frac{m_1+m_2-m_1}{m_1(m_1+m_2)}\right] = \frac{J^2 m_2}{2(m_1+m_2)m_1}$$

Now, the maximum compression of the spring is obtained when all of this energy is "converted" to PE in the spring

$$PE = \frac{1}{2}kx^2 \Rightarrow \text{Therefore} \quad H_0 = \frac{k x_{max}^2}{2} \quad \text{OR} \quad x_{max} = \sqrt{\frac{2H_0}{k}}$$

(2)

So,

$$x_{max} = \sqrt{\frac{2M_0}{K}} = \sqrt{\frac{2J^2 m_2}{2(m_1+m_2)m_1 K}} = J\sqrt{\left(\frac{m_2}{m_1}\right)\left(\frac{1}{M}\right)\left(\frac{1}{K}\right)}$$, where $M = m_1 + m_2$

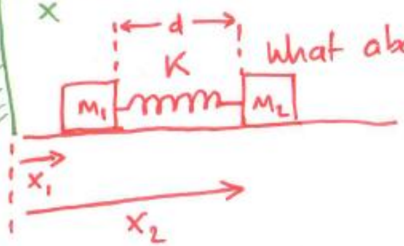
Now, you can imagine that the motion of the system after impact will include an oscillation (conservative forces acting), superimposed on a traslation of the C.O.M.

↳ of course, the question now is what will be the oscillation period of the system $T_0 = \frac{2\pi}{\omega}$

↳ what is ω_0 for 2 masses connected by a spring.

one mass has eqn. motion $m_1 \frac{d^2x}{dt^2} = -K(x-x_0)$ => solution $x(t) = X_0\cos(\omega_0 t + \varphi)$

$$\Rightarrow \omega_0 = \sqrt{\frac{K}{m_1}}.$$



What about 2 masses?

↳ Write eqns of motion for each mass

$$m_1 \frac{d^2x_1}{dt^2} = -KX \quad (A)$$

$X = x_2 - x_1 - d$ "COMPRESSION" OF SPRING

$$m_2 \frac{d^2x_2}{dt^2} = +KX \quad (B)$$

Multiply (A) × m_2 and (B) × m_1 and subtract.

$$m_2 m_1 \frac{d^2x_1}{dt^2} = +Km_2X \qquad m_1 m_2 \frac{d^2x_2}{dt^2} = \mp K m_1 X$$

$$\Rightarrow \quad m_1 m_2\left(-\frac{d^2x_2}{dt^2} + \frac{d^2x_1}{dt^2}\right) = Kx(m_1+m_2) \quad \Rightarrow \quad \frac{m_1 m_2}{(m_1+m_2)}\frac{d^2x}{dt^2} = -KX$$

If you write $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$ then $\mu = \frac{m_1 m_2}{m_1 + m_2}$ μ - is a reduced mass

$\Rightarrow$ obtain $\frac{d^2x}{dt^2} = -\left(\frac{k}{\mu}\right)x$ ie SHO with $\omega_0 = \sqrt{\frac{k}{\mu}}$

This is the natural frequency of our system

$\Rightarrow$ The motion of our system is described by.

$$x_1(t) = Vt + x_{max}\cos\left(\left[\sqrt{\frac{k(m_1+m_2)}{m_1 m_2}}\right]t + \varphi_1\right)$$

$$x_2(t) = Vt + x_{max}\cos\left(\left[\sqrt{\frac{k(m_1+m_2)}{m_1 m_2}}\right]t + \varphi_2\right) + d$$

$\bar{x}_1 + \bar{x}_2 = x_{max}$

ANS

Only 2 "Homework Heroes" this week !

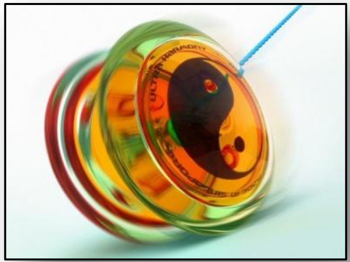
Lecture 4 - Contents

M4.1 Rotational *vs* Linear Dynamics

- *Dynamics of "rigid bodies" (starre Körper)...*
- *Torque (Drehmoment) and Lever arm (Hebelarm)...*
- *Energy of rotational motion...*
- *Angular momentum (Drehimpulse)...*
- *Newton´s laws for rotation...*

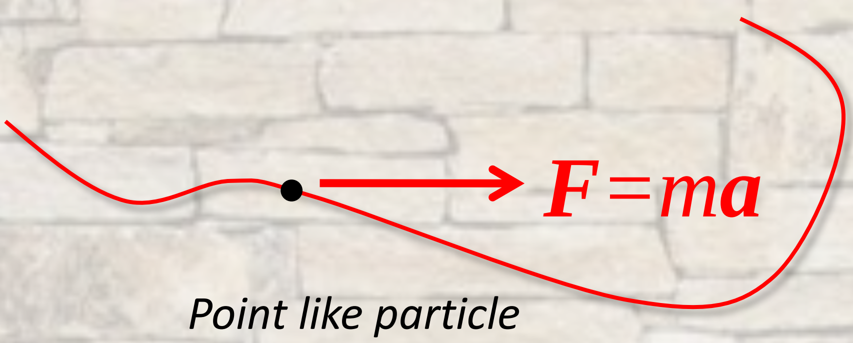
M4.2 Rotational motion of rigid bodies

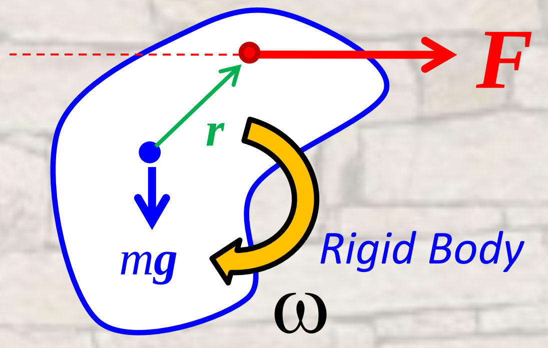
- *The moment of inertia* $\mathbf{I}$ *(Trägheitsmoment)...*
- *The parallel axis theorem...*
- *Angular precession and gyroscopes...*
- *Yo-Yo´s and angular momentum...*



4.1 Introduction to Rigid Body Dynamics

- Until now we've been considering the dynamics of **point like bodies** (e.g. elementary particles, point masses etc.)
  - Move along some trajectory in space in response to external forces
  - All forces act through the "center of mass" ($\mathbf{R_s}$) of the body
  - Some quantities (energy and linear momentum) are **constants of motion**





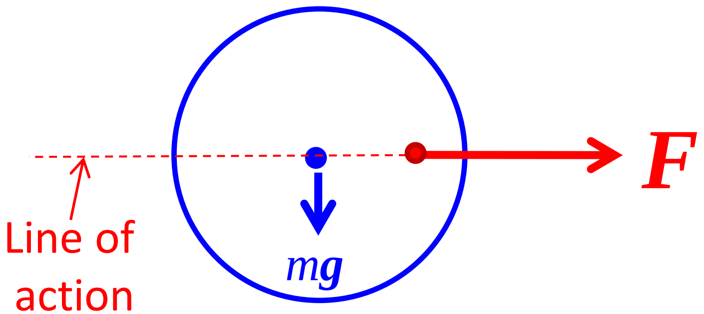
- The topic of rigid bodies (*starre Körper*) that we'll discuss today deals with the response of a non-deformable *extended* body to external forces
  - Forces do *not-necessarily* act through R_s
  - We have to consider the **rotation** of the body as well as **translation**

4.1.1 Torque

(Das Drehmoment)

- The motion of any rigid body is a combination of linear and rotational dynamics.
  - To see : consider the response of a circular disk to a force F with various “lines of action”

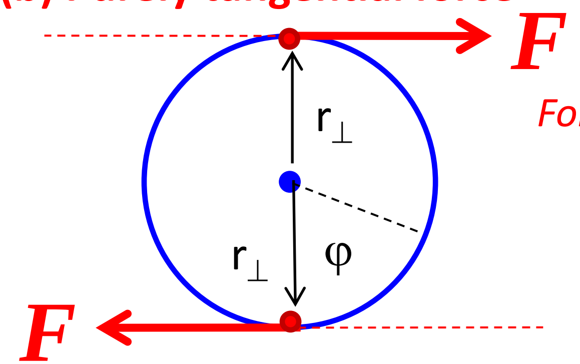
(a) Purely radial force



line of action passes through R_s

→ *Pure linear acceleration* a_s

(b) Purely tangential force



For a given force F, this becomes larger when ***Lever Arm*** $r_\perp$ *(Hebelarm) becomes larger*

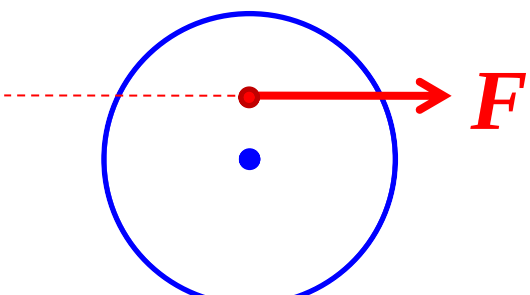
Purely tangential force pair, separated by $r_\perp$ from R_s

→*Zero linear acceleration since* $\Sigma \mathbf{F}=0$

→*Pure angular acceleration*

$$\dot{\omega} = \frac{d\omega}{dt} = \frac{d^2\varphi}{dt^2} = \ddot{\varphi}$$

(c) General case



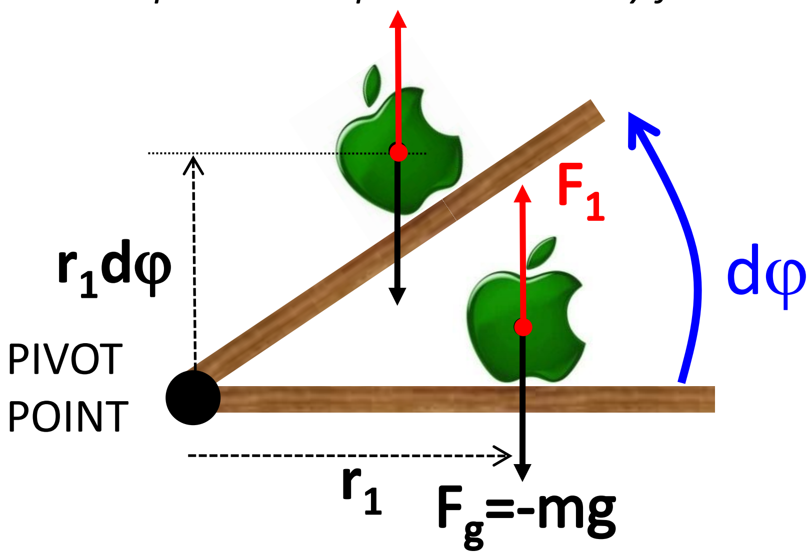
→ Linear and angular acceleration $\quad \dot{\omega} \neq 0 \quad \vec{a}_s \neq \vec{0}$

*We are used already to the fact that forces cause linear acceleration (**F**=ma)*

Apparently ***something related to F and*** $r_\perp$ ***can cause angular acceleration ?***

To prove some of these “common sense” ideas, consider an apple glued to a plank of wood

Pivot point – can provide arbitrary force at the point of rotation to fix that point to a rotational axis

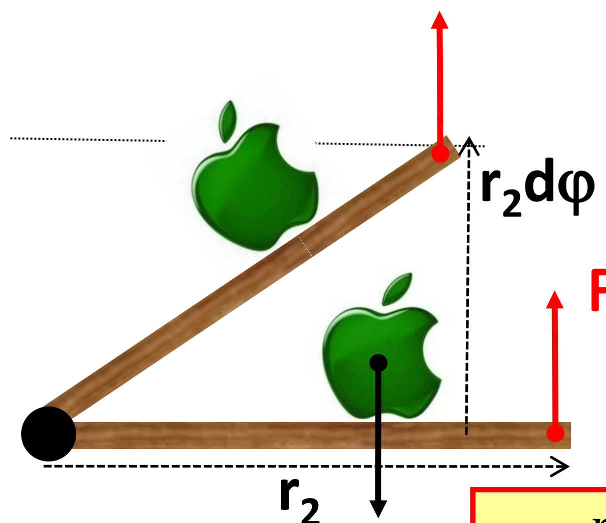


⇐Apply just enough force $\mathbf{F_1}$ at distance $\mathbf{r_1}$ from pivot point to balance gravity

This means that (Newton 3) $\vec{F}_1 = -\vec{F}_g$

If we now rotate the apple through angle $d\varphi$ then the work done dW is

$$dW = F_1 r_1 d\varphi$$



⇐If you now repeat the experiment with a force F_2 a distance r_2 from the pivot point to balance gravity and raise to same final state

What force need be applied ?

→ Total work done must be the same, since the final state is same and gravity conservative force

$$dW_1 = F_1(r_1 d\varphi) = dW_2 = F_2(r_2 d\varphi)$$

LEVER ARM (Hebelarm) RULE

$$F_1 = \frac{r_2}{r_1} F_2$$

$$F_1 r_1 = F_2 r_2 \quad = M$$

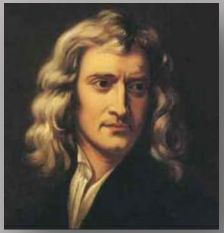
Define as **TORQUE** *(Drehmoment)*

Torque M is a force producing angular acceleration

→It´s a "twisting force"

If a body does not experience any linear acceleration then the net force acting on it is zero

$$\Sigma F=ma=m(dv/dt)$$

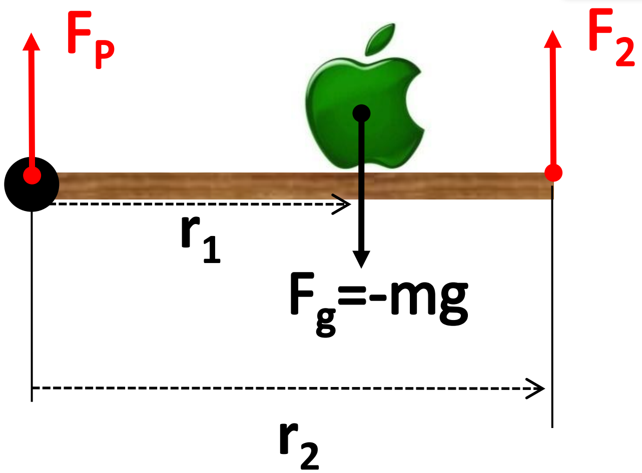


If a body does not experience any angular acceleration then the net torque acting on it is zero

$$\Sigma M \propto (d\omega/dt) \ ?$$

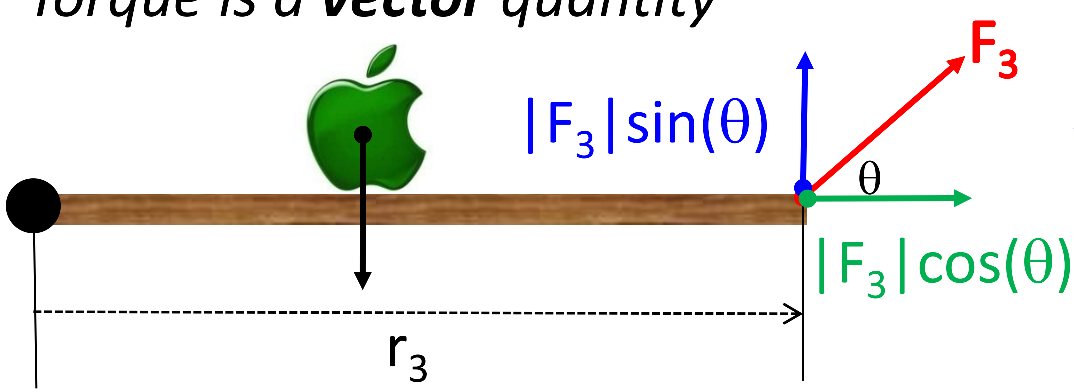
What are the total forces and torques acting in our apple problem ?

→ *The whole system is stationary, so total torque is zero (torque from pivot, gravity, F_2)*



TORQUE	FORCES
$M_p + M_g + M_2 = 0$	$F_p + F_g + F_2 = 0$
$(0)F_p + r_1 F_g + r_2 F_2 = 0$	$F_p - mg + \frac{r_1}{r_2} mg = 0$
$r_1 F_g = -r_2 F_2$	$F_p = +mg\left[\frac{r_2 - r_1}{r_2}\right]$
Torque provided by gravity equal + opposite to that from F_2	*Pivot force needed to balance torques*

*Torque is a **vector** quantity*

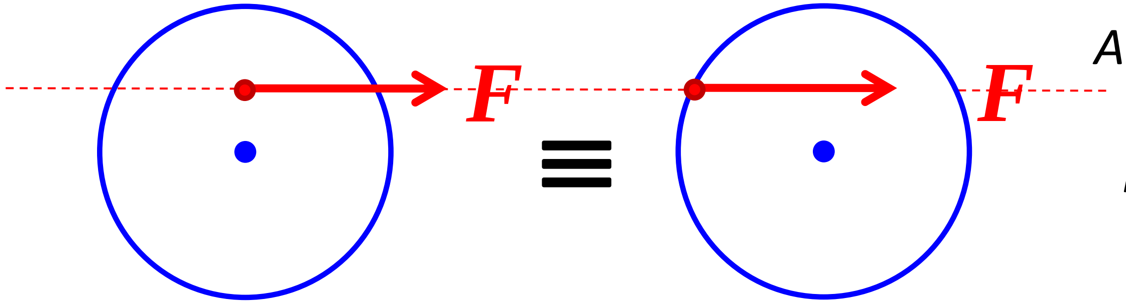


*If force $\mathbf{F_3}$ is applied at a point $\mathbf{r_3}$, at an angle relative to the radial vector $\mathbf{r_3}$ from the pivot point, only the tangential component of the force is relevant for the torque **M***

$$\left|\vec{M}\right| = \left|\vec{r}_3\right|\left|\vec{F}_3\right|\sin(\theta)$$

$$\vec{M} = \vec{r} \times \vec{F}$$

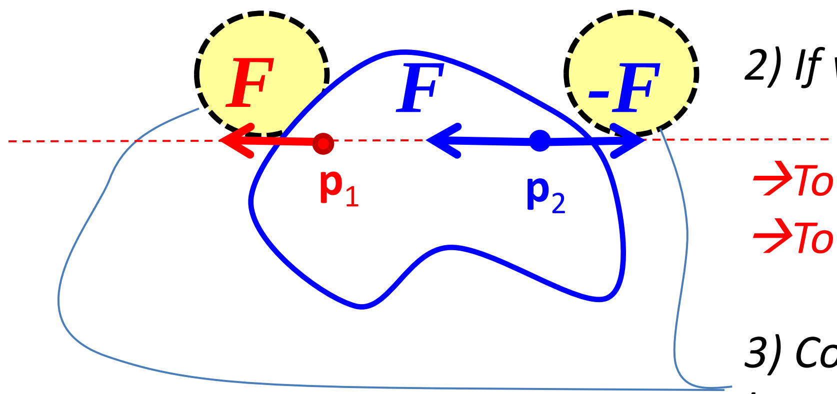
TORQUE M – vector product of radial vector and force acting



*A force **F** has the same effect when it acts along the line of action, irrespective of the precise position where it acts*

Proof

1) Consider a crazy shaped rigid body, with a force ***F_1*** *acting at point* ***p_1***



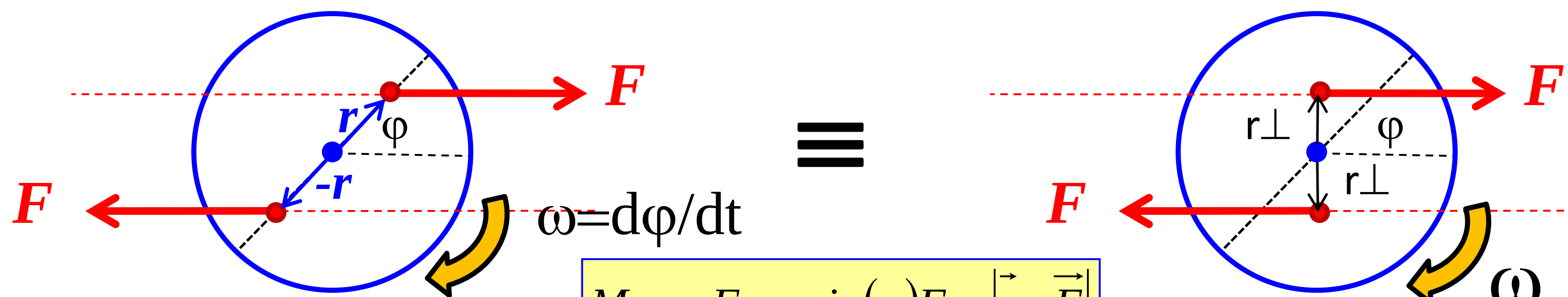
2) If we add a new *<u>force pair</u>* *at another point* ***p_2***

*→Total force **F** acting on the body is unchanged*

*→Total torque **M** acting on the body is unchanged*

*3) Could remove these forces, and situation is identical, but with Force **F** translated from point* ***p_1*** *to* ***p_2***

We can now **prove** that **M** is given by the *vector product* of **r** and **F**

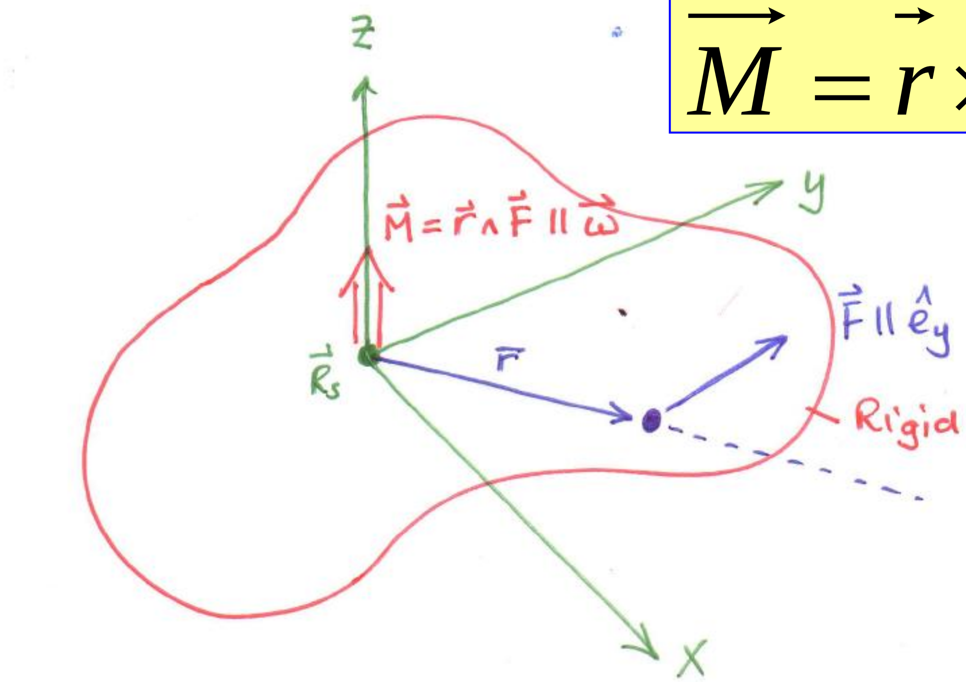


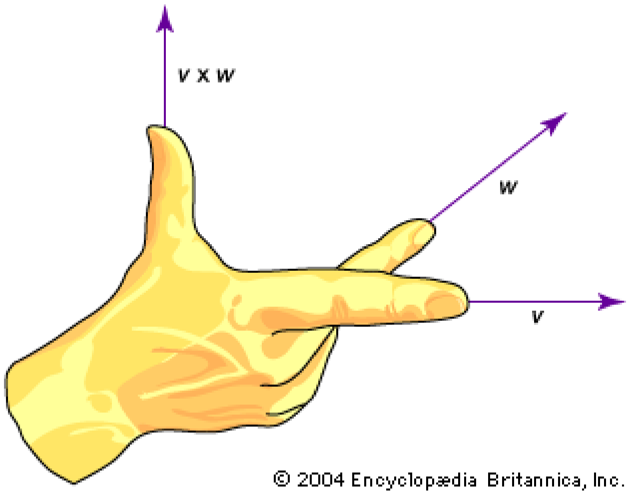
$$M = r_{\perp} F = r \sin(\varphi) F = \left| \vec{r} \times \vec{F} \right|$$

Direction of torque ?

- By convention, the vectors **r**, **F** and **M** define a right handed coordinate system

$$\vec{M} = \vec{r} \times \vec{F}$$





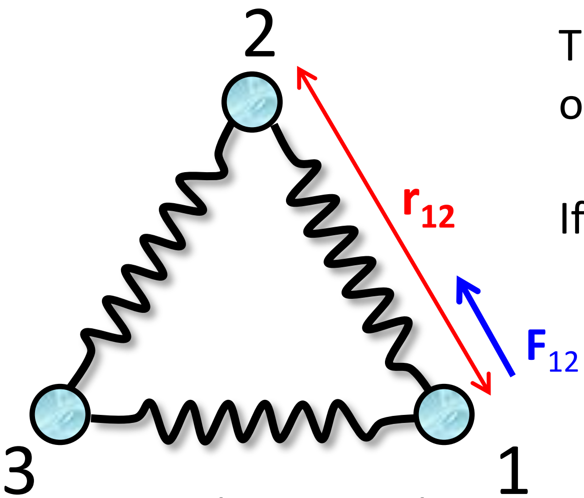
- **M** points in positive $\mathbf{e}_z$ direction when the resulting rotation (and angular acceleration $d\omega/dt$) is anticlockwise – by definition

- $\boldsymbol{\omega}$ and **M** are both axial vectors, whilst **r** and **F** are polar vectors

4.1.2 Energy of a rotating rigid body?

- Anyone who has tried to stop a rotating bicycle wheel with their hand knows that a rotating rigid body has energy (kinetic + potential)
  - We can express this rotational KE in terms of the angular velocity ω in the same way the translational $KE_{trans.} = \frac{1}{2} mv^2$ → we hope to get something like $KE_{rot} \propto \omega^2$

- To develop this idea a little, think of a rigid body as being made up of a large number of masses connected together by "stiff springs", such that $k \rightarrow \infty$
  - We are going to think of 3 "atoms" but the arguments apply to many atoms in a real rigid body!



The force acting between each mass is given by the compression or expansion (Δr) of the springs.

If we allow the restoring forces F_{12} etc. be finite then $\vec{F}_{12} = -k\Delta\vec{r}_{12}$

where $\Delta\vec{r}_{12} = \lim_{k\to\infty} \frac{-\vec{F}_{12}}{k} = 0$ i.e. displacements from ≡m spring length go to zero

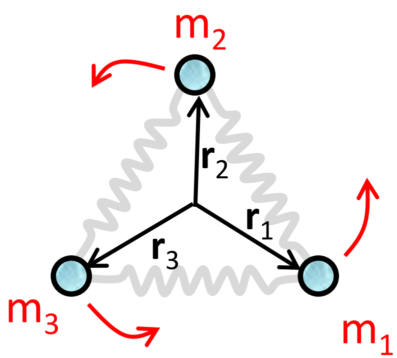
How much potential energy is stored "internally" in the springs ?

$$E_{pot} = \lim_{k\to\infty} \int_0^{\Delta r_{12}} -\vec{F}_{12}.d\vec{r}_{12} = \lim_{k\to\infty} \int_0^{F_{12}} + \frac{\left|\vec{F}\right|_{12}^{2}}{k} dF_{12} = \lim_{k\to\infty} \frac{\left|\vec{F}\right|_{12}^{3}}{3k} = 0$$

no PE is stored "internally" in a Rigid Body

(That´s why it's RIGID !)

- If the potential energy stored "internally" in a rigid body is zero, then what form must the energy take ?



$$E_{tot} = \cancel{E_{pot}} + E_{kin}$$ a rotating **rigid body** in CM frame has only rotational KE

$$\Rightarrow \quad E_{tot} = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 + \tfrac{1}{2} m_3 v_3^2 = \underbrace{\tfrac{1}{2} m_1 (r_1\omega)^2 + \tfrac{1}{2} m_2 (r_2\omega)^2 + \tfrac{1}{2} m_3 (r_3\omega)^2}$$

since $v=\omega r$ for rotational motion around the center of mass

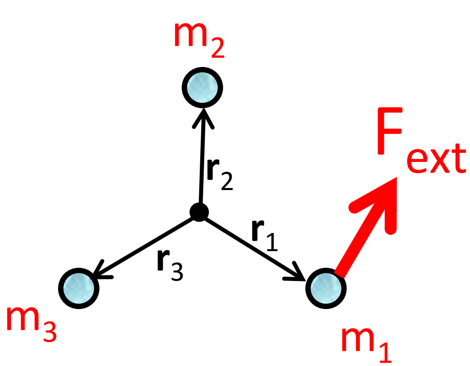
We then obtain $\Rightarrow$

$$E_{tot} = \tfrac{1}{2}\underbrace{\left(\sum_i m_i r_i^{\,2}\right)}\omega^2 = \tfrac{1}{2} I\omega^2$$

c.f. $$E_{KE,trans} = \tfrac{1}{2} M v^2$$

$$I = \left(\sum_i m_i r_i^{\,2}\right)$$ **MOMENT OF INERTIA** "plays role of mass"

Let the COM act now as a **pivot point** and consider the influence of an external tangential force on the rotational KE of the system...

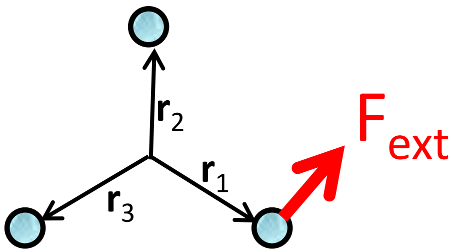


If individual forces were applied to each mass separately, they would obey

$$F_1 = m_1 r_1 \frac{d\omega}{dt} \quad , \quad F_2 = m_2 r_2 \frac{d\omega}{dt} \quad , \quad F_3 = m_3 r_3 \frac{d\omega}{dt}$$

BUT, since the tangential force is only applied at r_1, we must use the lever arm equation to find the effective forces acting at r_2 and r_3

In this case, when we take lever arm in account, we have for the tangential forces...



$$F_1 = F_{ext} = m_1 r_1 \frac{d\omega}{dt} \;, \quad F_2 = \left(\frac{r_1}{r_2}\right) F_{ext} \;, \quad F_3 = \frac{r_1}{r_3} F_{ext}$$

So, if F_{ext} were acting on each mass separately, we get

$$F_{ext} = m_1 r_1 \frac{d\omega}{dt} \;\overset{or}{,}\; F_{ext} = m_2 \frac{r_2^{\,2}}{r_1} \left(\frac{d\omega}{dt}\right) \;\overset{or}{,}\; F_{ext} = m_3 \frac{r_3^{\,2}}{r_1} \left(\frac{d\omega}{dt}\right)$$

This means that when F_{ext} acts on the entire system

$$F_{ext} = m_1 r_1 \left(\frac{d\omega}{dt}\right) + m_2 \frac{r_2^{\,2}}{r_1} \left(\frac{d\omega}{dt}\right) + m_3 \frac{r_3^{\,2}}{r_1} \left(\frac{d\omega}{dt}\right)$$

External torque exerted by the force on the whole system is then

$$M_{ext} = r_1 F_{ext} = m_1 r_1^{\,2} \left(\frac{d\omega}{dt}\right) + m_2 r_2^{\,2} \left(\frac{d\omega}{dt}\right) + m_3 r_3^{\,2} \left(\frac{d\omega}{dt}\right)$$

$$\Rightarrow \quad M_{ext} = \left(\sum_i m_i r_i^{\,2}\right)\left(\frac{d\omega}{dt}\right) = I\left(\frac{d\omega}{dt}\right) \qquad \boxed{M_{ext} = I\dot{\omega}}$$

EXTERNAL TORQUE
= MOM. OF INERTIA x ANGULAR ACC.

Angular momentum (L) - *conserved*

We have just shown that the externally applied torques to a rigid body (M_{ext}) are equal to the rate of change of a quantity $I\omega = L$

$$M_{ext} = \frac{d}{dt}(I\omega) = \frac{dL}{dt} \Rightarrow M_{ext} = 0 \Rightarrow \frac{dL}{dt} = 0 \quad \text{or} \quad L = const$$

$$L = I\omega$$

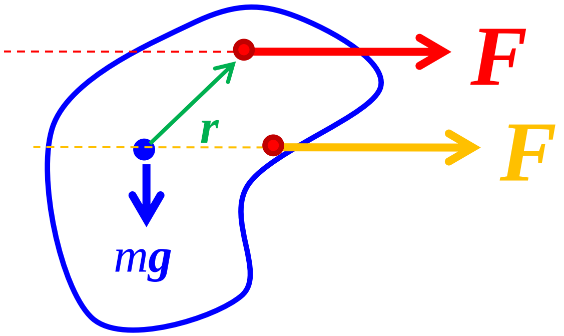
The angular momentum $L = I\omega$ is a conserved quantity of rotational motion when no or external torques M_{ext} are applied

↓ *compare with*

The linear momentum $\mathbf{p} = m\mathbf{v}$ is a conserved quantity of translational motion when no external forces F_{ext} are applied

Summary

Fundamentals of rotational motion



- A force with a line of action that passes through the COM of a rigid body creates only translational motion
- A force with a line of action that does *not* pass through the COM creates both translational and rotational motion
- We describe rotational motion of a rigid body using torques **M** to represent the action of forces **F** on it

$$\vec{M} = \vec{r} \times \vec{F}$$

- The analogy to "mass" for rigid bodies is the moment of inertia **I**

$$I = \sum_i m_i r_i^2$$

- The "internal energy" of a rigid body is only rotational

$$E = \tfrac{1}{2} I \dot{\omega}^2$$

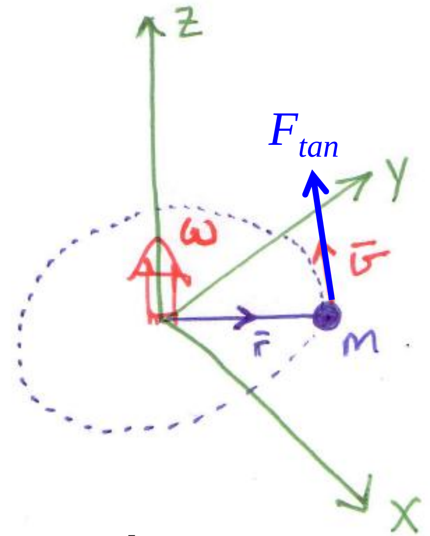
- Angular momentum L is conserved when no external torques act on the rigid body

$$M = \frac{d(I\omega)}{dt} = \frac{dL}{dt}$$

4.1.3 Newton´s Laws for Rotating Bodies

- We are now ready to develop some more fundamental relations for the rotational dynamics of a rigid body.
  - → We are going to show that the angular acceleration of a rotating rigid body (dω/dt) is proportional to the sum of the torque components along the axis or rotation Σ**M**
  - → The constant of proportionality between Σ**M** and dω/dt is the **moment of inertia I**

To do this, we are going to consider the simplest rotating system - a point mass moving on a circular path.



Velocity (**v**) related to angular velocity (**ω**) by $\vec{v} = \vec{\omega} \times \vec{r}$ ⟹ $v = \omega r$

Since $\vec{p} = m\vec{v}$, we can write $\vec{p} = m\left(\vec{\omega} \times \vec{r}\right)$ ⟹ $p = m\omega r$

A constant tangential force F_{tan} would result in an acceleration a

⟹ $F_{\tan} = ma = \frac{dp}{dt} = m\left(\frac{d\omega}{dt}\right)r$ ⟹ (x r) $rF_{\tan} = \left|\vec{r} \times \vec{F}\right| = \left|\vec{M}\right| = M$ $\qquad \frac{\partial r}{\partial t} = 0$

⟹ $rF_{\tan} = M = r\frac{dp}{dt} = \frac{d}{dt}(rp) = mr^2 \frac{\partial \omega}{\partial t}$

$M = \frac{dL}{dt}$ where $L = rp$ is the angular momentum

$M = I\left(\frac{\partial \omega}{\partial t}\right)^2$ where I=mr^2 is the moment of inertia

$M = \frac{dL}{dt}$ where $L = rp$ is the angular momentum | $M = I\left(\frac{\partial\omega}{\partial t}\right)^2$ where $I=mr^2$ is the moment of inertia

These equations are very similar to Newton´s 2nd law "multiplied by r"

$$F = \frac{dp}{dt} \xrightarrow{\times r} Fr = \frac{d}{dt}(rp) \rightarrow M = \dot{L}$$

$$F = m\dot{v} \xrightarrow{\times r} rF = mr^2\dot{\omega} \rightarrow M = I\dot{\omega}$$

For this special case of circular motion of a mass point, a number of nice analogies exist between translational and rotational dynamics

Translational Dynamics			**Rotational Dynamics**	
Position	r	(Spatial coordinate)	φ	(Drehwinkel)
Velocity	$v=dr/dt$	(Geschwindigkeit)	$\omega=d\varphi/dt$	(Winkelgeschwindigkeit)
Accel.	$a=dv/dt=d^2r/dt^2$	(Beschleunigung)	$d\omega/dt=d^2\varphi/dt^2$	(Winkelbeschleunigung)
Force	$F=ma$	(Kraft)	$M=rF$	(Drehmoment, Torque)
Momentum	$p=mv$	(Impuls)	$L=rp=rmv$	(Drehimpuls, Angular mom.)
Mass	m	(Masse)	$I=mr^2$	(Trägheitsmoment, Moment of Inertia)

*We can arrive at the same conclusions, but now remembering that **M, F, ω, r, v** etc are all **vectors***

Start at Newton´s 2nd

$$\vec{F} = m\vec{a} = \frac{d\vec{p}}{dt} \quad \xrightarrow{\mathbf{r}\times} \quad \vec{r}\times\vec{F} = \vec{r}\times\frac{d\vec{p}}{dt} = \vec{M}$$

$$\vec{M} = \frac{d}{dt}\left(\vec{r}\times\vec{p}\right) = \frac{d\vec{r}}{dt}\times\vec{p} + \vec{r}\times\frac{d\vec{p}}{dt} = \vec{r}\times\frac{d\vec{p}}{dt} = \frac{d\vec{L}}{dt}$$

*We have now defined the vector relationships for the angular momentum **L** and Torque **M***

$$\vec{L} = \vec{r}\times\vec{p}$$

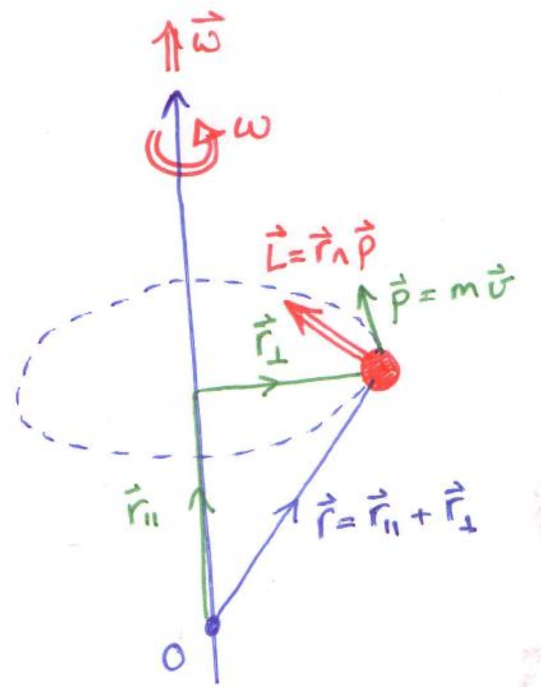
Angular momentum

Units $[L]=ML^2T^{-1}$=Nms=Js

$$\vec{M} = \vec{r}\times\vec{F} = \frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\vec{r}\times\vec{p}\right)$$

Torque

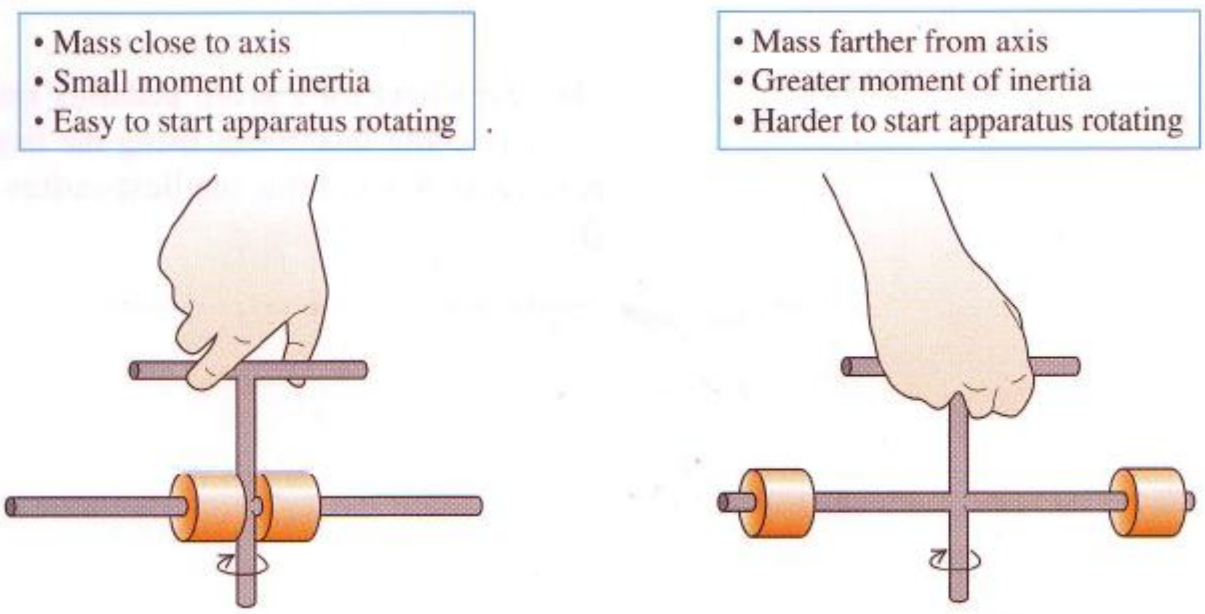
Units [M]=Nm=J



4.2 We've seen how the Moment of Inertia behaves like the "mass" for rotational motion

$$I = \sum_i m_i r_i^2$$

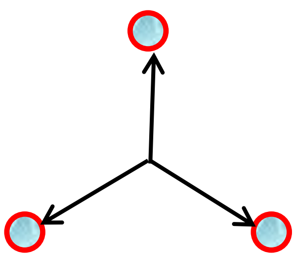
It kind of seems sensible that it should depend not just on the mass, but how it is distributed relative to the axis of rotation



We are now going to calculate the moment of inertia of some simply systems

4.2.1 Calculating moments of inertia

- Most rigid bodies are not discrete, i.e. represented by a few point masses, but consist of a continuous distribution of mass in space.
  - *The sum of masses and distances that defines the moment of inertia becomes an integral over mass elements*

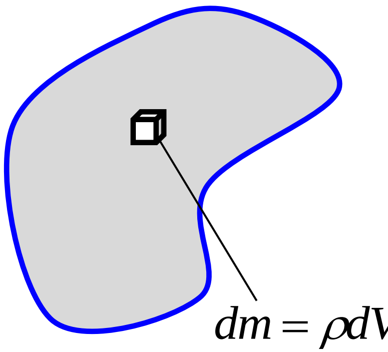


$$I = \sum_i m_i r_i^2$$

Discrete

$$I = \int r^2 dm$$

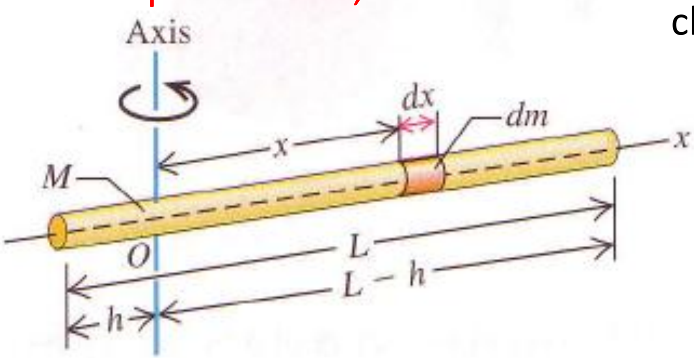
Continuous



We can describe the distribution of mass via it´s mass density ρ=dm/dV $I = \int r^2 \rho dV$

For a uniform mass density we can write $I = \rho \int r^2 dV$

Example : a thin, uniform bar

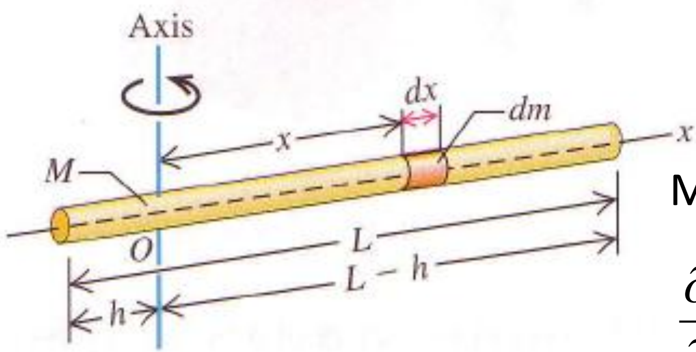


choose a volume element with length dx → $dm = A\rho dx$

$\rho = \frac{M}{AL}$

$$I = \int x^2 dm = \frac{M}{L} \int_{-h}^{L-h} x^2 dx = \frac{M}{L} \left(\frac{x^3}{3} \right) \Bigg|_{-h}^{L-h}$$

$$I = \frac{1}{3} M \left[L^2 - 3Lh + 3h^2 \right]$$ Ans.



$$I = \frac{1}{3}M\left[L^2 - 3Lh + 3h^2\right]$$

Max or minimum of I ?

$$\frac{\partial I}{\partial h} = \frac{1}{3}M\left[-3L + 6h\right] = 0$$

$$\frac{\partial^2 I}{\partial h^2} = 2M > 0$$

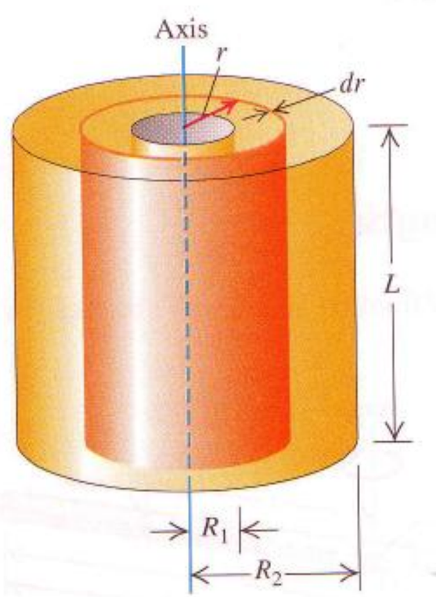
i.e. minimum

$$h = \frac{L}{2}$$

L – minimum when bar spins around its center of mass

Example 2 : Uniform disc with a radius *R*, thickness L via a rotation axis through its center

Easiest here to divide up the disc into infinitesimal cylindrical shells, thickness dr



Infinitesimal volume $dV = 2\pi r L dr$ Mass $dm = \rho dV = 2\pi r \rho L dr$

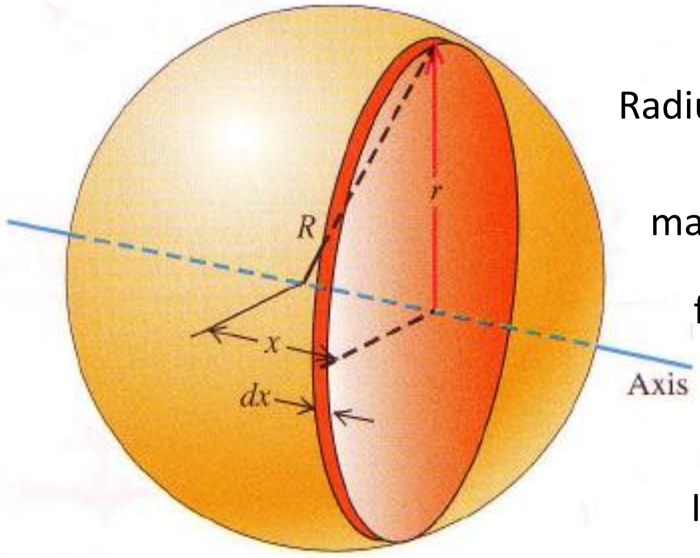
This infinitesimal shell contributes $dI = r^2 dm = r^2\left[2\pi r \rho L\right]dr$

$$I = \pi\rho L \int_{R_1}^{R_2} r^3 dr = \frac{\pi\rho L}{2}\left[R_2^4 - R_1^4\right] = \frac{\pi\rho L}{2}\left(R_2^2 - R_1^2\right)\left(R_2^2 + R_1^2\right)$$

Mass density $\rho = \dfrac{M}{\pi\left(R_2^2 - R_1^2\right)L}$ ➡ $I = \dfrac{1}{2}M\left(R_2^2 + R_1^2\right)$ Ans.

Without central hole $I = \dfrac{1}{2}MR^2$

Example 3 : Uniform sphere with a radius *R* via a rotation axis through its center



Easiest here to divide up the sphere into infinitesimal discs of thickness dx

Radius of disc $r = \sqrt{R^2 - x^2}$ Its volume $dV = \pi r^2 dx = \left(R^2 - x^2\right)dx$

mass $dm = \rho dV = \rho\pi\left(R^2 - x^2\right)dx$

for a solid disk with radius r and mass M=dm, we just showed that

$$dI = \tfrac{1}{2} r^2 dm = \tfrac{1}{2}\left(R^2 - x^2\right)\left[\pi\rho\left(R^2 - x^2\right)dx\right]$$

Integrating over whole sphere

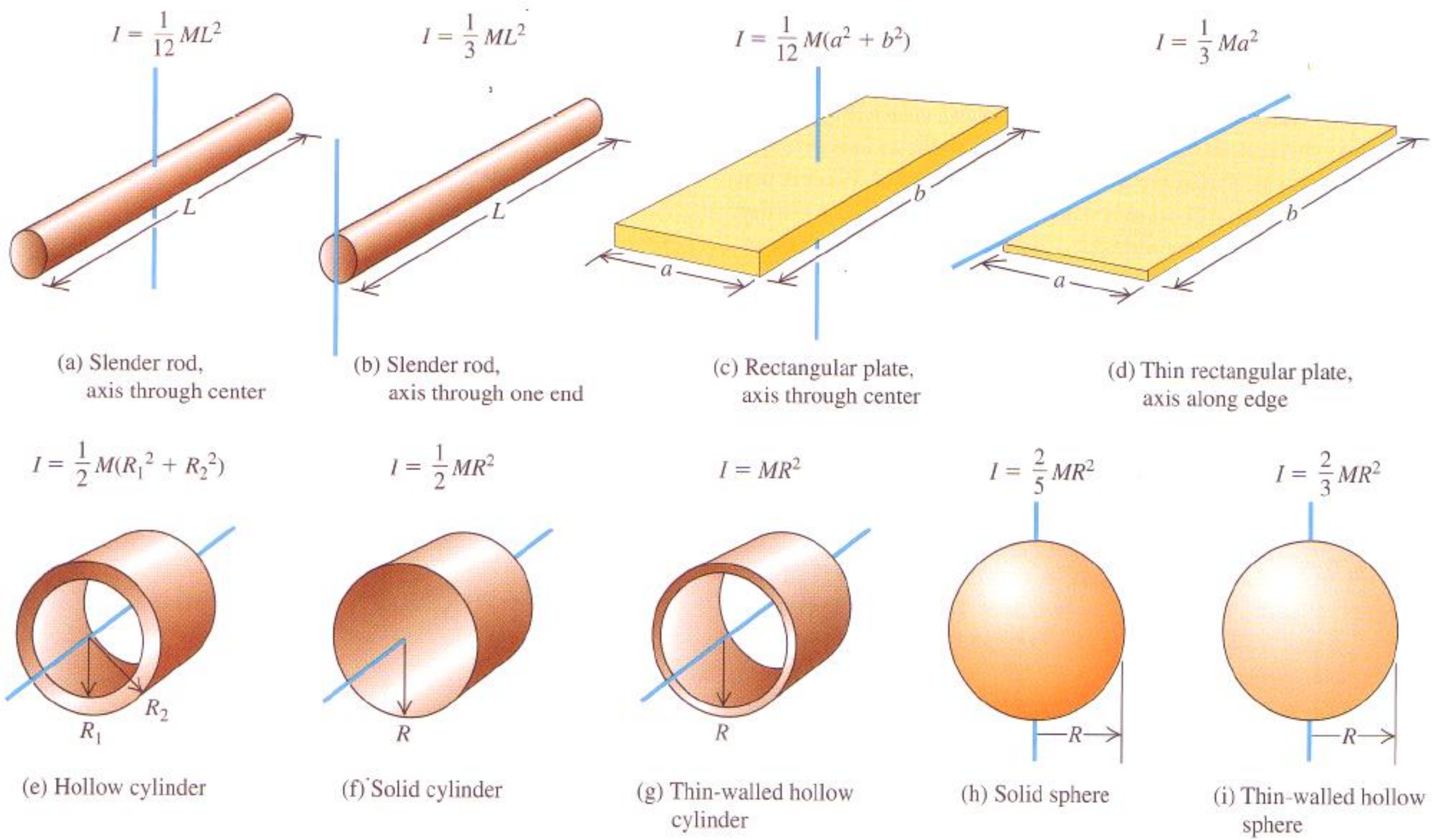
$$I = \tfrac{1}{2}\pi\rho \int_{-R}^{R} \left(R^2 - x^2\right)^2 dx = \frac{8\pi\rho}{15} R^5$$

Now we need the mass density ρ $M = \rho V = \frac{4\pi\rho R^3}{3}$

$$\rho = \frac{3M}{4\pi R^3}$$

We then obtain the answer ➡ $I_{sphere} = \frac{2}{5} MR^2$ Ans.

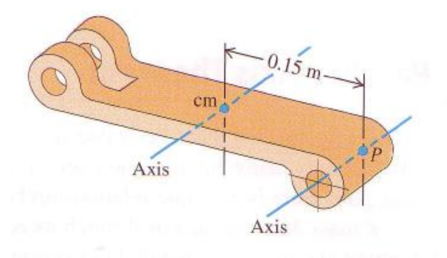
Some other commonly encountered examples



The parallel axis theorem (*Steinerscher Satz*)

- The moment of inertia of a rigid body depends on the distribution of mass around the axis of rotation

Problem → we have infinitely many axes of rotation !

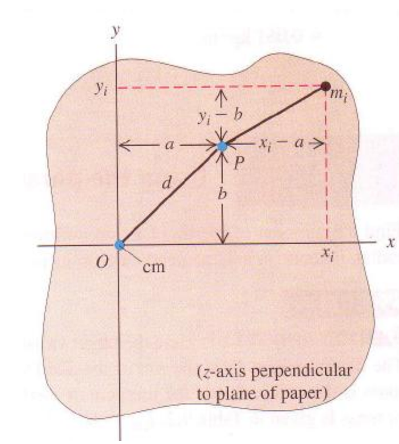


For example, I could take this mechanical linkage and have it rotate around an axis through its center of mass (cm) or another axis parallel to that and separated by 0.15m – the rotational energy is different for each

- There is a simple relationship between the moment of inertial about an arbitrary axis of rotation (I_P) and that passing through the center of mass (I_{CM})

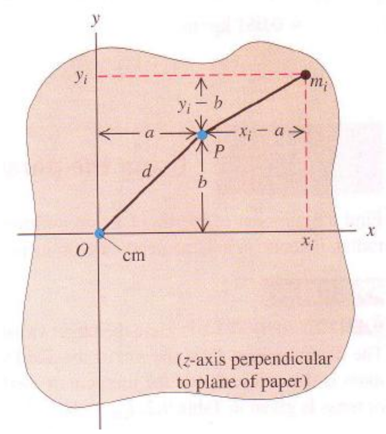
- The parallel axis theorem states → $I_P = I_{CM} + Md^2$

i.e. The moment of inertia around an arbitrary axis (p) is equal to the moment of inertia through the center of mass plus the CM moment relative to the original axis



Let´s prove this !

To prove the parallel axis theorem then consider the body shown below and two parallel axes of rotation **O** (through the C.O.M.) and **P** at $\mathbf{r}_P=(a,b,0)$



Clearly $d^2=a^2+b^2$ (1)

The moment of inertial about the axis through **O**

$$I_{CM}=\sum_i m_i\left(x_i^2+y_i^2\right) \quad (2)$$

The moment of inertial about the axis through **P**

$$I_P=\sum_i m_i\left(\left(x_i-a\right)^2+\left(y_i-b\right)^2\right) \quad (3)$$

These expressions don´t involve the co-ordinates z_i measured perpendicular to the slices, so we can extend the sums to include all particles in all slices

Expanding out 3 we get →

$$I_P=\underbrace{\sum_i m_i\left(x_i^2+y_i^2\right)}_{\text{From 2, this is } I_{CM}\text{ !}}\underbrace{-2a\sum_i m_i x_i-2b\sum_i m_i y_i}_{\text{Both terms are zero since x=0, y=0 is COM}}+\underbrace{\left(a^2+b^2\right)\sum_i m_i}_{\text{From 1, this is } Md^2}$$

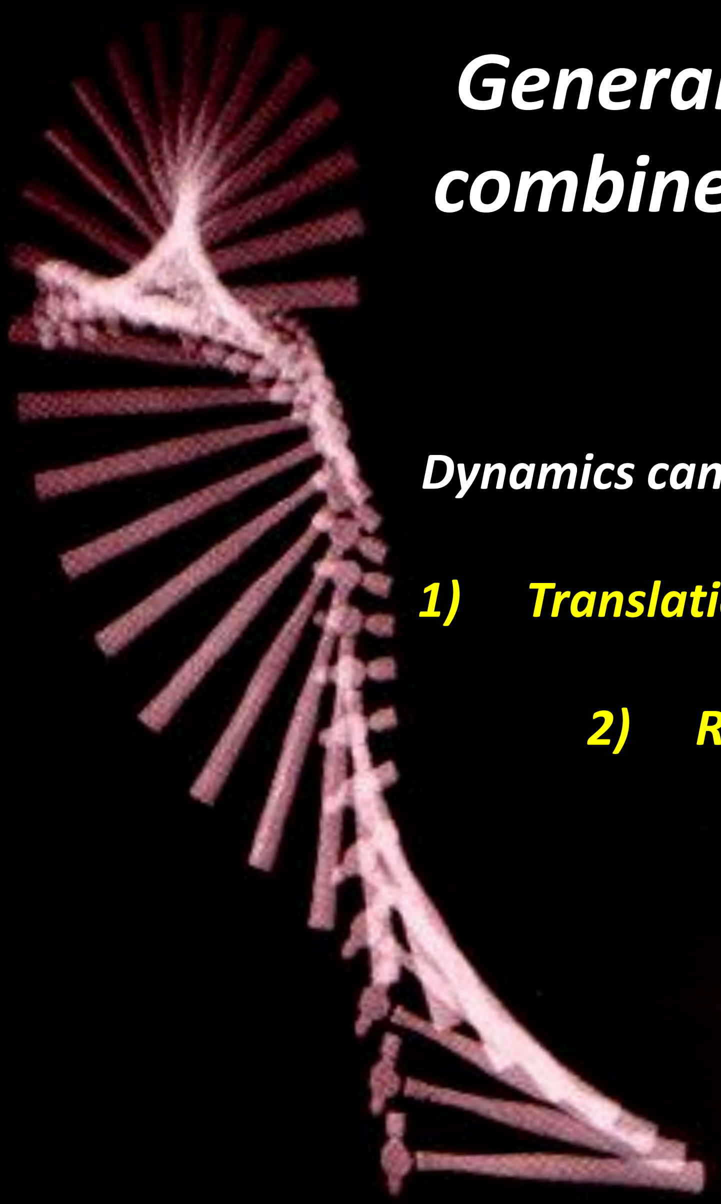
From 2, this is I_{CM} !

Both terms are zero since x=0, y=0 *is* COM

From 1, this is Md^2

We finally obtain → $I_P=I_{CM}+Md^2$ *proving* the parallel axis theorem

would indicate that it requires less work ($E_{KE\text{-}rot}=\frac{1}{2}I\omega^2$) to get an object rotating around its COM, compared with any other axis of rotation...



General motion of an extended body combines rotational and translational dynamics...

Dynamics can always be described as "two separate" motions

1) ***Translational motion of the COM as if it was a point mass***

2) ***Rotation around an axis through the COM***

This is a general theorem

Time lapse photography of throwing a hammer

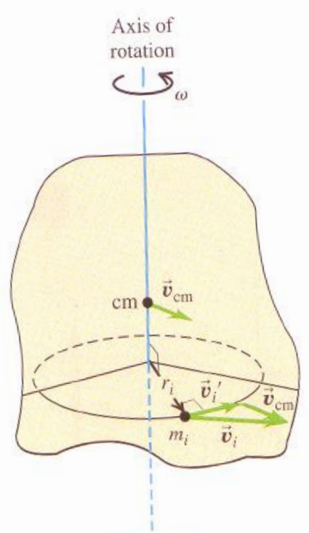
Rigid body rotation about a moving axis

- *We can extend our analysis of the dynamics of rotational motion to the general case when the axis of rotation can move (translate) in space*

"every possible motion of a rigid body can be represented as a combination of translational motion of the center of mass and rotation around the axis of the center of mass"

Let's show that this is true for the kinetic energy of a rigid body with rotational and translational motion :

$$E_{KE} = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$$



Velocity $\vec{v}_i$ of particle in rotating, translating rigid body = (velocity $\vec{v}_{cm}$ of center of mass) plus (particle's velocity $\vec{v}_i'$ relative to center of mass)

1)Remember that the rigid body is made up of i particles that are distributed in space and each of which is moving with a velocity $\boldsymbol{v}_i$

$$\vec{v}_i = \vec{v}_{CM} + \vec{v}'_i$$

velocity of COM

velocity of i[th] particle relative to COM

2)The KE of the i^{th} particle in the inertial frame is ½ mv_i^2, so

$$K_i = \frac{1}{2}m_i\left(\vec{v}_i.\vec{v}_i\right) = \frac{1}{2}m_i\left(\left(\vec{v}_{cm} + \vec{v}'_i\right).\left(\vec{v}_{cm} + \vec{v}'_i\right)\right)$$

3)The total KE of the *entire body* is then $E_{KE} = \sum_i K_i = \frac{1}{2}\sum_i m_i\left(v_{cm}^2 + 2\vec{v}_{cm}.\vec{v}'_i + v_i'^2\right)$

We then obtain

$$E_{KE} = \frac{1}{2}\left(\sum_i m_i\right)v_{cm}^2 + \vec{v}_{cm}.\sum_i m_i\vec{v}'_i + \frac{1}{2}\sum_i m_i v_i'^2$$

$$E_{KE} = \frac{1}{2}\left(\sum_i m_i\right) v_{cm}^2 + \vec{v}_{cm} \cdot \sum_i m_i \vec{v'}_i + \frac{1}{2}\sum_i m_i v'^2_i$$

$$\frac{1}{2} M v_{cm}^2$$

Translational KE of COM

$$\frac{1}{2}\sum_i m_i v'^2_i$$ =Rotational energy

To see this, remember $v = \omega r$

$$\frac{1}{2}\sum_i m_i v'^2_i = \frac{1}{2}\omega^2 \sum_i m_i r'^2_i = \frac{1}{2} I_{cm}\omega^2$$

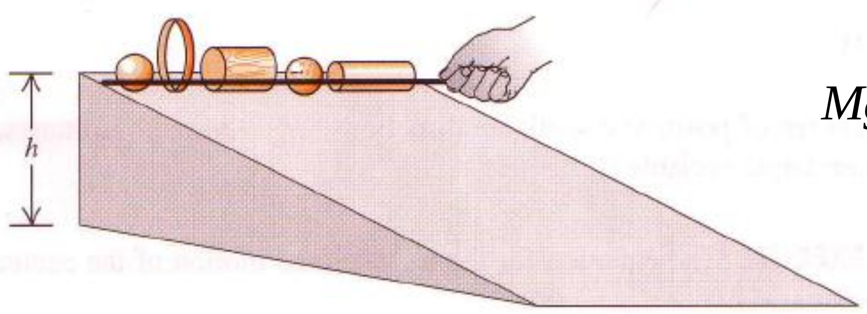
This summation must go to zero since it is equal to M times the velocity of the center of mass, *relative to the center of mass* → zero by definition

$$E_{KE} = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{CM}\omega^2$$

General kinetic energy of a rotating + translating body....

Example - which body makes it faster down the slope ?

Energy conservation $K_{top} + U_{top} = K_{bottom} + U_{bottom}$



$$Mgh = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I\omega^2$$

$$I = cMR^2$$

$$\omega = \frac{v_{cm}}{R}$$

$$Mgh = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} cMR^2 \frac{v_{cm}^2}{R^2}$$

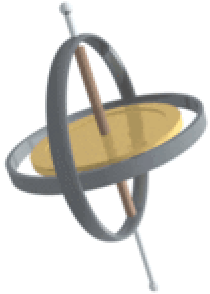
$$v_{cm} = \sqrt{\frac{2gh}{1+c}}$$

Independent of R and M!

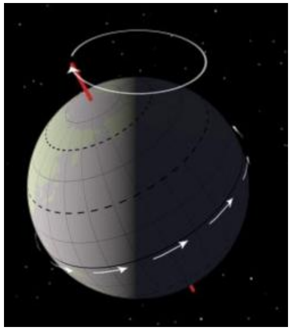
Huge cylinders have same speed as small ones

Angular precession

- Until now, we have only considered the situation when the direction of the axis of rotation remains fixed in space
  - Interesting, and rather unexpected things can happen when we try to change the direction of the **L**=Iω vector
  - Most important is angular precession, the gradual precession of **L** around another axis **Ω**



Gyroscope



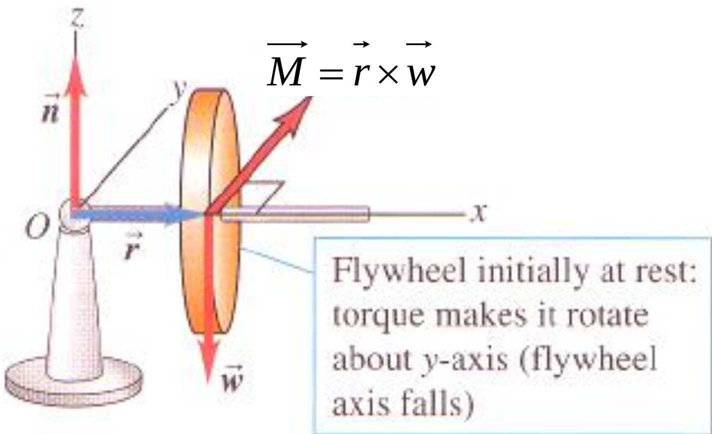
Earth precession (26000 year period)

Angular momentum causes strange things to happen

Precession occurs due to the relation between torque (**M**) and the change of angular momentum (d**L**/dt)

DISC NOT SPINNING

$\vec{M} = \vec{r} \times \vec{w}$



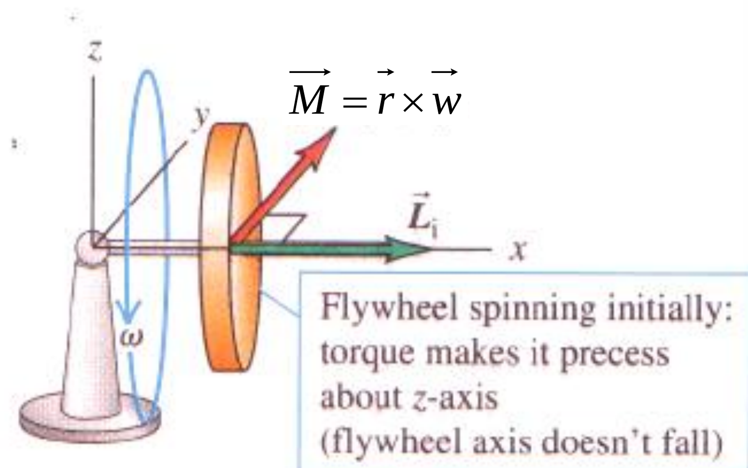
The gravitational force (**w=mg**) acts downwards

*1) If the flywheel (disc) is not spinning, then the disc has no angular momentum (**L**=Iω=**0**, since ω=**0**)*

*Gravity force (**w**) produces a torque **M**, that causes the flywheel to fall down.*

DISC IS SPINNING

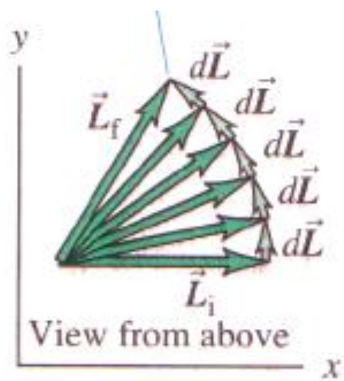
$\vec{M} = \vec{r} \times \vec{w}$



*2) The direction of **L** "tries" to change due to the torque **M** induced by the gravity force*

*But, a changing **L**, gives rise itself to a torque since* $\vec{M} = \dfrac{d\vec{L}}{dt}$

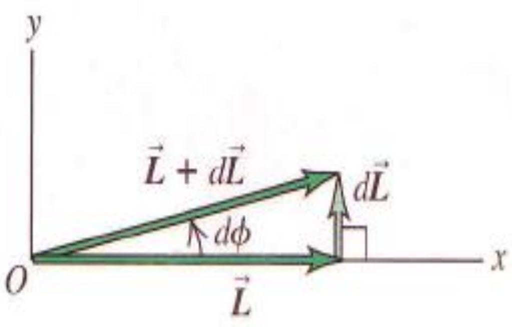
Therefore → $d\vec{L} = \vec{M}dt$



*There is an initial angular momentum **L**$_i$, torque **M** only changes the direction of L, but not it's magnitude*

*Since d**L** always || **M** and **M** ⊥ **L***

d**L** is always in the (x,y) plane, i.e. **L** precesses around the z-axis but does not fall down



At any instant in time t the gyroscope has angular momentum $\boldsymbol{L}$

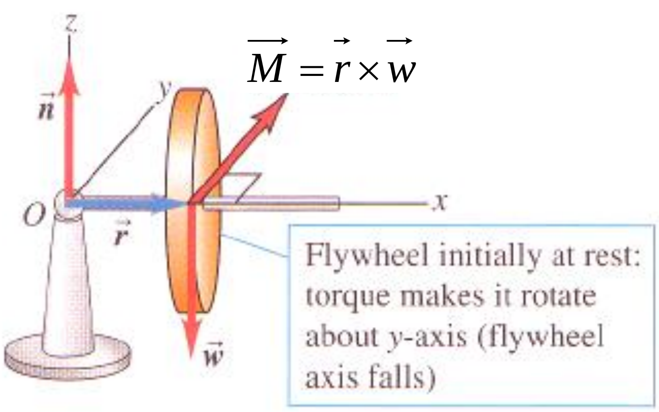
A short time dt later it's angular momentum has changed to $\boldsymbol{L}+d\boldsymbol{L}$

The direction of the angular momentum vector has precessed through an angle dϕ as shown on the vector diagram left.

$$\tan(d\phi)=\frac{|d\vec{L}|}{|\vec{L}|} \xrightarrow{\text{small angle}} d\phi \approx \frac{|d\vec{L}|}{|\vec{L}|}$$

Precession angular speed is

$$\Omega=\frac{d\phi}{dt}=\frac{|d\vec{L}|/|\vec{L}|}{dt}=\frac{|\vec{M}|}{|\vec{L}|}=\frac{wr}{I\omega}$$



$$\Omega=\frac{rmg}{I\omega}$$

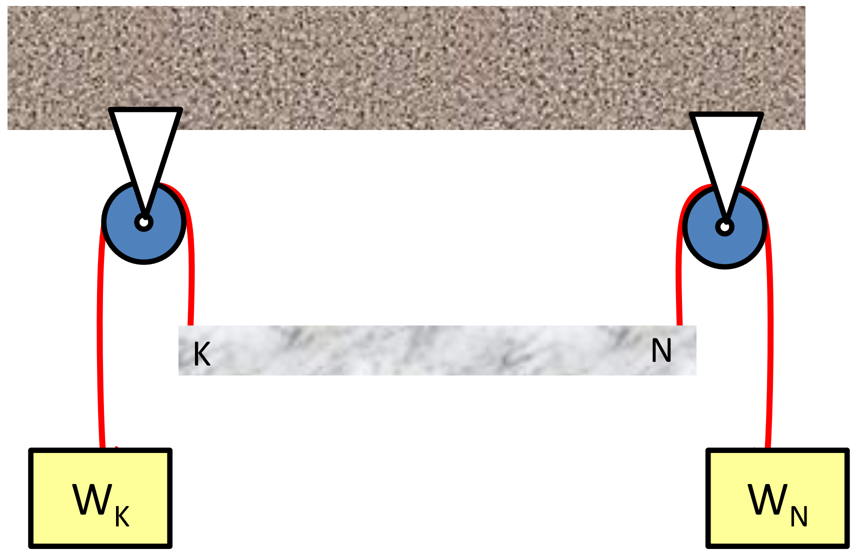
Gyroscope precesses faster as ω reduces !

The gyroscope precession frequency (Ω) should <u>speed up</u> as friction causes it to slow down

Wobble upon slowing down ?

Homework 4

The figure below shows two pulleys holding a rigid bar K-N and two weights (W_K and W_N) in equilibrium



The string holding the system at point N is suddenly cut. Given the length L and mass m of the bar

Q) Find the initial acceleration of (a) end K and (b) end N of the bar