Model-Preserving Sensitivity Analysis for Families of Gaussian Distributions

Manuele Leonelli

Graphical Models: Conditional Independence and Algebraic Structures

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Joint work with C. Görgen

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Plan for the Talk

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Why Sensitivity Analysis?

- The accuracy of probability distributions inferred using machine-learning algorithms heavily depends on data availability and quality.
- In practical applications it is therefore fundamental to investigate the robustness of a statistical model to misspecification of some of its underlying probabilities.
- In the context of graphical models, investigations of robustness fall under the notion of *sensitivity analyses*.
- These analyses consist in varying some of the model's probabilities or parameters and then assessing how far apart the original and the varied distributions are.

Gaussian Independence Models

- Let Y be a n-dimensional Gaussian random vector with mean $\mu \in \mathbb{R}^n$, covariance matrix $\Sigma \in \mathbb{R}^{n \times n}_{\text{spsd}}$ and $f_{\mu,\Sigma}$ its density.
- For $A, B \subseteq [n] = \{1, \ldots, n\}$, let $\mu_A = (\mu_i)_{i \in A}$ and $\Sigma_{A,B}$ be the submatrix of Σ with rows indexed by A and columns indexed by B.
- For any two disjoint sets $A, B \subset [n]$, $Y_A = (Y_i)_{i \in A}$ has density $f_{\mu_A, \Sigma_{A,A}}$ and $Y_A | Y_B = y_B$ has density $f_{\mu^{A|B}, \Sigma^{A|B}}$ where

$$\mu^{A|B} = \mu_A + \Sigma_{A,B} \Sigma_{B,B}^{-1} (y_B - \mu_B)$$

$$\Sigma^{A|B} = \Sigma_{A,A} - \Sigma_{A,B} \Sigma_{B,B}^{-1} \Sigma_{B,A}.$$

Gaussian Independence Models

The random vector Y_A is said to be *conditionally independent of* Y_B *given* Y_C for disjoint subsets $A, B, C \subseteq [n]$ if and only if the density factorizes as

$$f_{\mu^{A\cup B|C},\Sigma^{A\cup B|C}} = f_{\mu^{A|C},\Sigma^{A|C}} f_{\mu^{B|C},\Sigma^{B|C}}.$$

and write $A \perp\!\!\!\perp B \mid C$.

Drton et al. 2008

For a n-dimensional Gaussian random vector Y with density $f_{\mu,\Sigma}$ and disjoint $A, B, C \subset [n]$, the conditional independence statement $A \perp\!\!\!\perp B \mid C$ is true if and only if all $(\#C+1) \times (\#C+1)$ minors of the matrix $\Sigma_{A\cup C,B\cup C}$ are equal to zero. Here, $\#C$ denotes the cardinality of the set C.

Gaussian Independence Models

- Let $\text{CI} = \{A_1 \perp\!\!\!\perp B_1 \mid C_1, \ldots, A_r \perp\!\!\!\perp B_r \mid C_r\}$ for disjoint index sets $A_i, B_i, C_i \subset [n]$ and $i \in [r]$, with $r \in \mathbb{N}$.

- A *Gaussian conditional independence model* $\mathcal{M}_{\text{CI}}$ for which all CI statements are true is a special subset of all possible Gaussian densities $f_{\mu,\Sigma}$:

$$\mathcal{M}_{\text{CI}} \subseteq \{f_{\mu,\Sigma} \mid \mu \in \mathbb{R}^n, \Sigma \in \mathbb{R}^{n\times n}_{\text{spsd}}\}.$$

- The parameter space of $\mathcal{M}_{\text{CI}}$ is equal to the algebraic set

$$\mathcal{A}_{\text{CI}} = \{\mu \in \mathbb{R}^n, \Sigma \in \mathbb{R}^{n\times n}_{\text{spsd}} \mid g(\Sigma) = 0 \text{ for all polynomials } g$$

which are $(\#C_i{+}1)\times(\#C_i{+}1)$ minors of $\Sigma_{A_i\cup C_i, B_i\cup C_i}, i \in [r]\}$.

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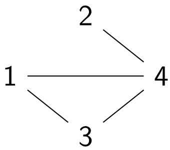
Undirected Graphical Models

A *Gaussian undirected graphical model* for a random vector $Y = (Y_i)_{i\in[n]}$ is defined by an undirected graph $\mathcal{G} = (V, E)$ with vertex set $V = [n]$ and a family of densities $f_{\mu,\Sigma}$ whose covariance matrix Σ is such that $(\Sigma^{-1})_{ij} = 0$ if and only if $(i, j) \notin E$.

The statement $Y_2 \perp\!\!\!\perp \{Y_1, Y_3\} \mid Y_4$ can be represented by the undirected graph



The 2×2 minors of the submatrix

$$\Sigma_{\{2,4\},\{1,3,4\}} = \begin{pmatrix} \sigma_{21} & \sigma_{23} & \sigma_{24} \\ \sigma_{41} & \sigma_{43} & \sigma_{44} \end{pmatrix}$$

need to vanish. Explicitly,
$\sigma_{21}\sigma_{43} - \sigma_{41}\sigma_{23} = 0$, $\sigma_{21}\sigma_{44} - \sigma_{41}\sigma_{24} = 0$ and $\sigma_{23}\sigma_{44} - \sigma_{43}\sigma_{24} = 0$.

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Gaussian Bayesian Networks

A *Gaussian Bayesian network* for a random vector $Y = (Y_i)_{i \in [n]}$ is a DAG $\mathcal{G} = (V, E)$ with $V = [n]$ and conditional Gaussian densities f_{μ_i, σ_i} with mean $\mu_i = \beta_{0i} + \sum_{j \in \mathrm{pa}(i)} \beta_{ji} y_j$ and variance $\sigma_i \in \mathbb{R}_+$, with $\mathrm{pa}(i) \subseteq [i-1]$.

- Conditional independences $Y_i \perp\!\!\!\perp Y_{[i-1] \setminus \mathrm{pa}(i)} \mid Y_{\mathrm{pa}(i)}$
- Define
  - $\beta_0 = (\beta_{0i})_{i \in [n]}$ the vector of intercepts
  - B be the strictly upper triangular matrix with entries $B_{ji} = \beta_{ji}$ if $j \in \mathrm{pa}(i)$ and zero otherwise
  - $\Phi = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ be the diagonal matrix of the conditional variances
- Then Y has Gaussian density $f_{\mu, \Sigma}$ with mean $\mu = (I - B)^{-\top} \beta_0$ and covariance matrix $\Sigma = (I - B)^{-\top} \Phi (I - B)^{-1}$

Gaussian Bayesian Networks

Consider $Y_3 \perp\!\!\!\perp Y_1 \mid Y_2$

$$1 \longrightarrow 2 \longrightarrow 3$$

the 2×2 minors of the submatrix

$$\Sigma_{\{2,3\},\{1,2\}} = \begin{pmatrix} \sigma_{21} & \sigma_{22} \\ \sigma_{31} & \sigma_{32} \end{pmatrix}$$

need to vanish. Here the only vanishing minor simply corresponds to the determinant. So $g = \sigma_{21}\sigma_{32} - \sigma_{31}\sigma_{22}$ is a polynomial which must be zero.

Sensitivity Analysis for Gaussian Models

- For a generic Gaussian random vector Y with density $f_{\mu,\Sigma}$, robustness is usually studied by perturbing the mean vector μ and the covariance matrix Σ.
- Such a perturbation is carried out by defining a perturbation vector $d \in \mathbb{R}^n$ and a matrix $D \in \mathbb{R}^{n \times n}$ which act additively on the original mean and variance, giving rise to a vector $\tilde{Y}$ with a new distribution $f_{\mu+d,\Sigma+D}$.
- The dissimilarity between these two vectors is then usually quantified via the KL divergence.

$$\mathrm{KL}(\tilde{Y}||Y) = \frac{1}{2}\left(\mathrm{tr}(\Sigma^{-1}D) + d^\top \Sigma^{-1} d + \ln\left(\frac{\det(\Sigma)}{\det(\Sigma + D)}\right)\right).$$

What's the issue?

$$1 \longrightarrow 2 \longrightarrow 3$$

- Suppose D has all zeros except for a $d \in \mathbb{R}$ in positions $(2, 1)$ and $(1, 2)$ such that $\Sigma + D \in \mathbb{R}^{3\times 3}_{\text{spsd}}$.
- The graph is still valid if and only if the 2×2 minor $(\sigma_{21} + d)\sigma_{32} - \sigma_{31}\sigma_{22}$ is equal to zero.
- But this is the case if and only if $d = 0$: so if there is no perturbation.
- If alternatively the only non-zero entry of D were in position $(1, 1)$ then no matter what the value of $d \in \mathbb{R}$ the graph would be valid.

Possible Solutions

- Work with the conditional Gaussian distributions.
- Perturb the matrix Φ of conditional variances which then affects Σ.
- Perturb the matrix B of regression coefficients which then affects μ and Σ.
- However and critically, both these approaches lose the intuitiveness of acting directly on the unconditional mean and covariance of the Gaussian distribution.

Our Proposal

- Consider a Gaussian model $\mathcal{M}_{\mathrm{CI}}$ for a random vector $Y = (Y_i)_{i \in [n]}$ together with conditional independence assumptions $\mathrm{CI} = \{A_k \perp\!\!\!\perp B_k \mid C_k \text{ for } k \in [r]\}$ as being represented by a collection of vanishing minors of its covariance matrix $\Sigma \in \mathbb{R}^{n \times n}_{\mathrm{spsd}}$.
- Without loss suppose $\mu = 0_n$ and write f_Σ.
- Let

$$\Phi_\Delta : \Sigma \mapsto \Delta \circ \Sigma$$

  denote the map which sends a covariance matrix to its Schur product with a matrix Δ.
- We call the map Φ_Δ *model-preserving* if under this operation the algebraic parameter set is mapped onto itself, $\Phi_\Delta(\mathcal{A}_{\mathrm{CI}}) \subseteq \mathcal{A}_{\mathrm{CI}}$.

Variation and Covariation Matrices

We decompose the perturbation of a covariance matrix Σ into two steps, and hence two Schur products.

1. Σ is mapped to its Schur product with a symmetric *variation* matrix $\Delta \in \mathbb{R}^{n \times n}_{\neq 0}$. Some σ_{ij} are assigned a new value $\sigma_{ij} \mapsto \delta_{ij}\sigma_{ij}$ at selected positions (i, j) while all others are equal to one.

In demanding that all entries δ_{ij} are non-zero, we automatically avoid setting a non-zero covariance $\sigma_{ij} \neq 0$ to zero via multiplication by an entry of Δ. This type of perturbation would force the corresponding variables to be independent, $X_i \perp\!\!\!\perp X_j$, in the perturbed model, which would clearly violate the assumptions in the original model $\mathcal{M}_{\mathrm{CI}}$.

Variation and Covariation Matrices

We decompose the perturbation of a covariance matrix Σ into two steps, and hence two Schur products.

2. A Schur product between $\Delta \circ \Sigma$ and a symmetric *covariation* matrix $\tilde{\Delta} \in \mathbb{R}^{n \times n}_{\neq 0}$. This matrix $\tilde{\Delta}$ has ones in the positions (i, j) whilst the others are to be set to ensure model-preservation.

$$\tilde{\Delta} \circ \Delta \circ \Sigma = \begin{pmatrix} \star & \cdots & \cdots & \star \\ \vdots & \ddots & 1 & \vdots \\ \vdots & 1 & \ddots & \vdots \\ \star & \cdots & \cdots & \star \end{pmatrix} \circ \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & \ddots & \delta_{ij} & \vdots \\ \vdots & \delta_{ji} & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix} \circ \begin{pmatrix} \sigma_{11} & \cdots & \cdots & \sigma_{1n} \\ \vdots & \ddots & \sigma_{ij} & \vdots \\ \vdots & \sigma_{ji} & \ddots & \vdots \\ \sigma_{n1} & \cdots & \cdots & \sigma_{nn} \end{pmatrix}$$

We need to find $\tilde{\Delta}$ such that $\tilde{\Delta} \circ \Delta \circ \Sigma \in \mathcal{A}_{\mathrm{CI}}$. Then the map $\Phi_{\tilde{\Delta} \circ \Delta}$ is model-preserving.

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Example

Consider $Y_3 \perp\!\!\!\perp Y_2 \mid Y_1$ and perturb σ_{21} . Then

$$\Delta = \begin{pmatrix} 1 & \delta & 1 \\ \delta & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and the only vanishing minor of $\Delta \circ \Sigma$ takes the form $\delta\sigma_{12}\sigma_{32} - \sigma_{31}\sigma_{22}$. This polynomial is equal to zero in either of three cases

- σ_{22} is covaried by δ;
- σ_{31} and σ_{13} are covaried by δ
- σ_{22}, σ_{31}, σ_{13}, σ_{32} and σ_{23} are covaried by δ.

The associated covariation matrices $\tilde{\Delta}$ are,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \delta & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & \delta \\ 1 & 1 & 1 \\ \delta & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & \delta \\ 1 & \delta & \delta \\ \delta & \delta & 1 \end{pmatrix}.$$

For these $\tilde{\Delta}$, we have that $\Phi_{\tilde{\Delta}\circ\Delta}$ is model-preserving.
Consider $\tilde{\Delta}_{\{2,3\},\{1,2\}} \circ \Delta_{\{2,3\},\{1,2\}}$. Then the matrices are equal to

$$\begin{pmatrix} \delta & \delta \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} \delta & 1 \\ \delta & 1 \end{pmatrix}, \quad \begin{pmatrix} \delta & \delta \\ \delta & \delta \end{pmatrix}.$$

Some Notation

For any symmetric matrix $D \in \mathbb{R}^{n\times n}$ and two index sets $A, B \subseteq [n]$, we henceforth denote with $\lfloor D_{A,B} \rfloor^1$ the symmetric, full dimension $n \times n$ matrix where:

- all positions indexed by A and B are equal to the corresponding entries in D;
- entries not indexed by A and B are set to ensure symmetry;
- all other entries are equal to one.

Let $D \in \mathbb{R}^{3\times 3}$ and suppose

$$D_{\{1,2\},\{2,3\}} = \begin{pmatrix} 1 & \delta \\ 1 & \delta \end{pmatrix}.$$

Then

$$\lfloor D_{\{1,2\},\{2,3\}} \rfloor^1 = \begin{pmatrix} 1 & 1 & \delta \\ 1 & 1 & \delta \\ \delta & \delta & 1 \end{pmatrix}.$$

Covariation Matrices

For a single-parameter variation matrix Δ with $\delta_{ij} = \delta_{ji} = \delta$, we say that the covariation matrix $\tilde{\Delta}$ is

- *total* if $\tilde{\Delta} \circ \Delta = \delta \mathbb{1}_{[n],[n]}$;
- *partial* if $\tilde{\Delta} \circ \Delta = \lfloor \delta \mathbb{1}_{A \cup C, B \cup C} \rfloor^1$.
- *row-based* if $\tilde{\Delta} \circ \Delta = \lfloor \delta \mathbb{1}_{E, B \cup C} \rfloor^1$ for a subset $E \subseteq A \cup C$;
- *column-based* if $\tilde{\Delta} \circ \Delta = \lfloor \delta \mathbb{1}_{A \cup C, F} \rfloor^1$ for a subset $F \subseteq B \cup C$.

By construction total, partial, row- and column-based covariations ensure symmetry. Henceforth, we assume that the perturbed matrix $\tilde{\Delta} \circ \Delta \circ \Sigma$ is also positive semidefinite, so that $\tilde{\Delta} \circ \Delta \circ \Sigma \in \mathbb{R}^{n \times n}_{\text{spsd}}$.

One Independence Statement CI $= \{A \perp\!\!\!\perp B \mid C\}$

- If $(i,j),(j,i) \notin (A \cup C, B \cup C)$ then the map $\Phi_{\tilde{\Delta}\circ\Delta}$ is model-preserving for a covariation $\tilde{\Delta} = \mathbb{1}_{[n],[n]}$.
- If $C = \emptyset$ then the map $\Phi_{\tilde{\Delta}\circ\Delta}$ is model-preserving for $\tilde{\Delta} = \mathbb{1}_{[n],[n]}$.
- The map $\Phi_{\tilde{\Delta}\circ\Delta}$ is model-preserving for total and partial covariation matrices $\tilde{\Delta}$.

For total covariation matrices, $\delta > 0$. For partial covariations this may not have to be enforced, but it is rare to investigate the effect of changing the sign. Furthermore, increasing interest on covariance matrices whose entries are positive.

One Independence Statement CI $= \{A \perp\!\!\!\perp B \mid C\}$

The map $\Phi_{\tilde{\Delta} \circ \Delta}$ is model-preserving in the following cases:

- if (i,j) or $(j,i) \in (A,B)$ for a row-based covariation $\tilde{\Delta}$ whenever $i \in E \subseteq A$, and for a column-based covariation $\tilde{\Delta}$ whenever $j \in F \subseteq B$;
- if (i,j) or $(j,i) \in (A,C)$ for a row-based covariation $\tilde{\Delta}$ whenever $i \in E \subseteq A$, and for a column-based covariation $\tilde{\Delta}$ whenever $F = C$;
- if (i,j) or $(j,i) \in (C,B)$ for a row-based covariation $\tilde{\Delta}$ whenever $E = C$, and for a column-based covariation $\tilde{\Delta}$ whenever $i \in F \subseteq B$;
- if (i,j) and $(j,i) \in (C,C)$ for a row-based covariation $\tilde{\Delta}$ whenever $E = C$, and for a column-based covariation $\tilde{\Delta}$ whenever $F = C$.

Multiple CI Statements

Consider $Y_4 \perp\!\!\!\perp Y_{\{1,2\}} \mid Y_3$ and $Y_{\{2,4\}} \perp\!\!\!\perp Y_5 | Y_3$. The associated submatrices are

$$\begin{pmatrix} \sigma_{31} & \sigma_{32} & \sigma_{33} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sigma_{23} & \sigma_{25} \\ \sigma_{33} & \sigma_{35} \\ \sigma_{43} & \sigma_{45} \end{pmatrix}.$$

Suppose the entry σ_{43} is perturbed by δ and pick two $\tilde{\Delta}$. Then

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \delta & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \delta & 1 & 1 \\ 1 & \delta & \delta & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & \delta & 1 \\ 1 & 1 & \delta & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \delta & 1 & 1 \\ 1 & \delta & \delta^2 & \delta & 1 \\ 1 & 1 & \delta & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Multiple CI Statements

Let $\text{CI} = \{A_1 \perp\!\!\!\perp B_1 \mid C_1, \ldots, A_r \perp\!\!\!\perp B_r \mid C_r\}$, $A = \cup_{k\in[r]} A_k$, $B = \cup_{k\in[r]} B_k$ and $C = \cup_{k\in[r]} C_k$.

- Standard conditional independences can be eliminated from CI;
- We introduced a notion of separable CIs for which we can define covariations independently
- total and partial covariations are model-preserving.

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Multiple CI Statements

Let $\text{CI} = \{A_1 \perp\!\!\!\perp B_1 \mid C_1, \dots, A_r \perp\!\!\!\perp B_r \mid C_r\}$, $A = \cup_{k\in[r]} A_k$, $B = \cup_{k\in[r]} B_k$ and $C = \cup_{k\in[r]} C_k$.

- Standard conditional independences can be eliminated from CI;
- We introduced a notion of separable CIs for which we can define covariations independently
- total and partial covariations are model-preserving.

The map $\Phi_{\tilde{\Delta}\circ\Delta}$ is model-preserving for a row-based or a column-based covariation matrix $\tilde{\Delta}$ if

$$\tilde{\Delta}_{A\cup C, B\cup C} \circ \Delta_{A\cup C, B\cup C} = (\lfloor \tilde{\Delta}_{A\cup C, B\cup C} \circ \Delta_{A\cup C, B\cup C} \rfloor^1)_{A\cup C, B\cup C}.$$

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Multi-way Sensitivity Analysis

Compositions of model-preserving maps are model-preserving. In particular, for any two matrices Δ and Δ' we have $\Phi_{\Delta}(\Phi_{\Delta'}) = \Phi_{\Delta \circ \Delta'}$.

- We can write $\Delta = \Delta^1 \circ \Delta^2 \circ \cdots \circ \Delta^n$ where every Δ^k enforces a single-parameter variation.
- We can covary every single-parameter variation Δ^k by a matrix $\tilde{\Delta}^k$ using any covariation.
- Because the Schur product is commutative, this induces a map

$$\Phi_{\tilde{\Delta}^1 \circ \Delta^1 \circ \tilde{\Delta}^2 \circ \Delta^2 \circ \cdots \circ \tilde{\Delta}^n \circ \Delta^n} = \Phi_{\tilde{\Delta}^1 \circ \tilde{\Delta}^2 \circ \cdots \circ \tilde{\Delta}^n \circ \Delta^1 \circ \Delta^2 \circ \cdots \circ \Delta^n} = \Phi_{\tilde{\Delta} \circ \Delta}$$

where $\tilde{\Delta} = \tilde{\Delta}^1 \circ \tilde{\Delta}^2 \circ \cdots \circ \tilde{\Delta}^n$ is the covariation matrix for Δ.

Divergence Quantification - KL Divergence

The KL divergence between Y and $\tilde{Y}$ in model-preserving sensitivity analyses can be written as

$$\mathrm{KL}(\tilde{Y}||Y) = \frac{1}{2}\left[\mathrm{tr}(\Sigma^{-1}(\tilde{\Delta}\circ\Delta\circ\Sigma)) - n + \log\frac{\det(\Sigma)}{\det(\tilde{\Delta}\circ\Delta\circ\Sigma)}\right].$$

For total covariation matrices KL divergence has the following simple closed-form formula.

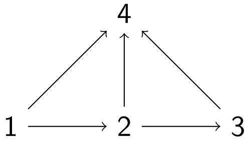
$$\mathrm{KL}(\tilde{Y}||Y) = \frac{1}{2}\left(n(\delta - \log(\delta) - 1)\right)$$

where $\delta = \prod_{i\in[n]}\delta_i$ for a multi-way variation.

Divergence Quantification - Frobenius Norm

- The *Frobenius norm* between zero-mean Gaussians Y and Y' is
$$\mathrm{F}(Y, Y') = \mathrm{tr}((\Sigma - \Sigma')^\top (\Sigma - \Sigma')).$$
- Let $\tilde{\Delta} \circ \Delta = (\delta_{ij})_{ij}$ be model-preserving. Then
$$\mathrm{F}(Y, \tilde{Y}) = \sum_{i,j \in [n]} (1 - \delta_{ij})^2 \sigma_{ij}^2. \tag{1}$$
- For standard sensitivity analyses $\mathrm{F}(Y, Y') = \mathrm{tr}(D^\top D)$.
- We can rank methods:
  - $\mathrm{F}(Y, \tilde{Y}_{\text{total}}) \geq \mathrm{F}(Y, \tilde{Y}_{\text{partial}})$
  - $\mathrm{F}(Y, \tilde{Y}_{\text{partial}}) \geq \mathrm{F}(Y, \tilde{Y}_{\text{row}})$
  - $\mathrm{F}(Y, \tilde{Y}_{\text{partial}}) \geq \mathrm{F}(Y, \tilde{Y}_{\text{column}})$
  - $\mathrm{F}(Y, \tilde{Y}_{\text{column}}) \geq \mathrm{F}(Y, \tilde{Y}_{\text{standard}})$
  - $\mathrm{F}(Y, \tilde{Y}_{\text{row}}) \geq \mathrm{F}(Y, \tilde{Y}_{\text{standard}})$

A First Example



$$\Sigma = \begin{pmatrix} 1 & 2 & 2 & 7 \\ 2 & 5 & 5 & 17 \\ 2 & 5 & 6 & 19 \\ 7 & 17 & 19 & 63 \end{pmatrix}.$$

One conditional independence statement, $Y_3 \perp\!\!\!\perp Y_1 \mid Y_2$, vanishing minor $\sigma_{12}\sigma_{23} - \sigma_{22}\sigma_{13} = 0$. Thus only variations of the parameters σ_{21}, σ_{22}, σ_{31} and σ_{32} may break the conditional independence structure of this model.

KL Divergence

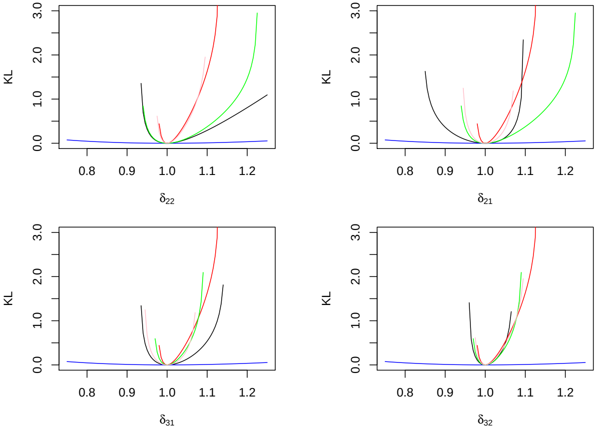


Figure: black = standard variation; blue = full; red = partial; green = row-based; pink = column-based.

Frobenius Norm

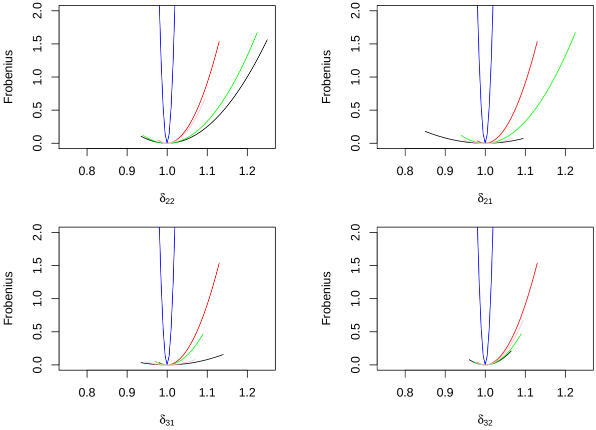


Figure: black = standard; blue = full; red = partial; green = row-based; pink = column-based.

Two-way Sensitivity

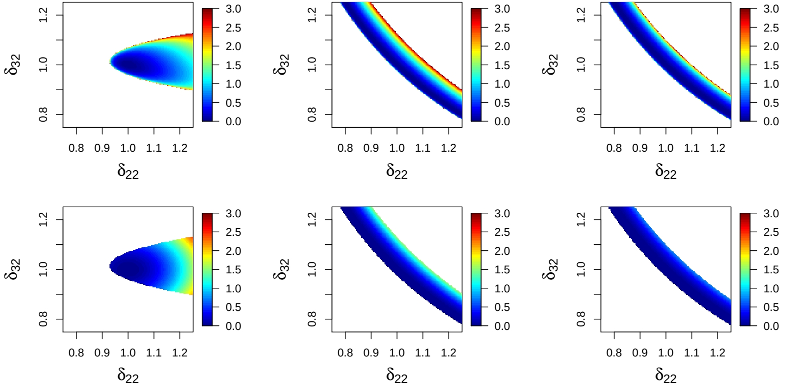
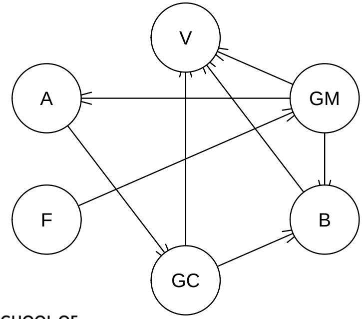


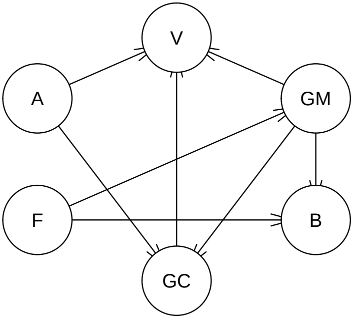
Figure: KL divergence (top) and Frobenius norm (bottom) for multi-way variation of the parametes σ_{22} and σ_{32}

Real-World Application

Metabolomic information of 77 individuals: 47 of them suffering of cachexia, whilst the remaining do not. Cachexia is a metabolic syndrome characterized by loss of muscle with or without loss of fat mass.

We focus on only six metabolics: Adipate (A), Betaine (B), Fumarate (F), Glucose (GC), Glutamine (GM) and Valine (V).





Real-World Application

	B	V	GC	GM	A	F
B	304	3262	220	2963	414	208
V	3262	98456	6637	89431	12489	6279
GC	220	6637	3950	53223	1693	839
GM	2963	89431	53223	3050126	65012	31858
A	414	12489	1695	65012	7279	1791
F	208	6279	839	31858	1791	1124

	B	V	GC	GM	A	F
B	41	1004	0	310	168	51
V	1004	38647	0	11923	10192	1974
GC	0	0	109	376	0	77
GM	310	11923	376	8952	3144	1092
A	168	10192	0	3144	5171	520
	51	1974	77	1092	520	192

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KL Divergence

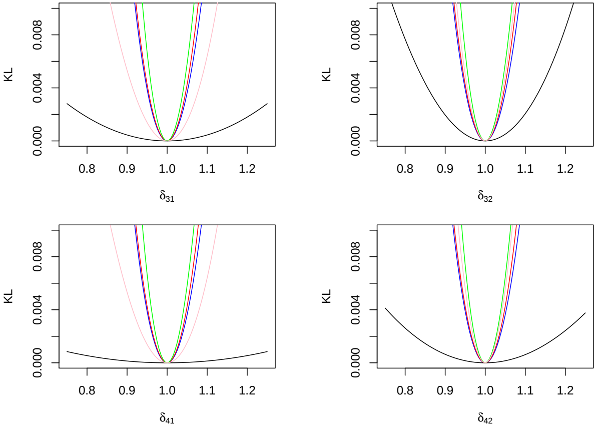

Figure: We use the color code black = standard variation; blue = full; red = partial; green = row-based; pink = column-based.

Frobenius Norm

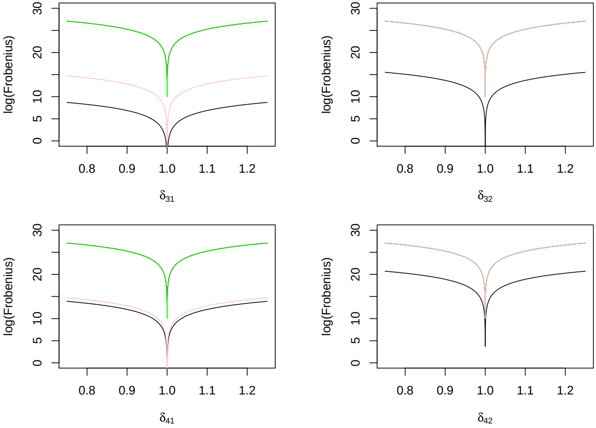

Figure: black = standard variation; blue = full; red = partial; green = row-based; pink = column-based.

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Conclusions

- A new approach to sensitivity analysis which does not break the conditional independence structure of the model;
- The effect of changing additional entries may (or may not) increase the divergence between the original and the varied distributions, depending on the form of the matrix
- For standard analyses, the theory of interval matrices can tell us for which variations the matrix is still positive-semidefinite
- An R package implementing these methods (as well as standard ones) is currently being developed.